An analysis of the models $L[T_{2n}]$

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Abstract

We analyze the models $L[T_{2n}]$, where T_{2n} is a tree on $\omega \times \kappa^1_{2n+1}$ projecting to a universal Π^1_{2n} set of reals, for $n > 1$. Following Hjorth's work on $L[T_2]$, we show that under $\mathrm{Det}(\underset{\sim}{\Pi}^1_{2n})$, the models $L[T_{2n}]$ are unique, that is they do not depend of the choice of the tree T_{2n}. This requires a generalization of the Kechris-Martin theorem to all pointclasses Π^1_{2n+1}. We then characterize these models as constructible models relative to the direct limit of all countable non-dropping iterates of $\mathcal{M}^{\#}_{2n+1}$. We then show that the GCH holds in $L[T_{2n}]$, for every $n < \omega$, even though they are not extender models. This analysis localizes the HOD analysis of Steel and Woodin at the even levels of the projective hierarchy.

1 Introduction

The purpose of this paper is to study the inner models $L[T_{2n}]$ associated with the pointclasses $\underset{\sim}{\Pi}^1_{2n}$. The study of the inner models $L[T]$ where T is a tree on $\omega \times \kappa$, κ a Suslin cardinal, T projects to a universal Γ set of reals and Γ is a Levy pointclass, was initiated in the works of the Cabal. The first step in this direction was taken by Moschovakis who has shown that if T_1 is the Schoenfield tree on $\omega \times \omega_1$ projecting to a Π^1_1 set of reals then

$$L[T_1] = L.$$

In [13], Moschovakis defined the models H_Γ as follows. Fix a pointclass Γ which resembles Π^1_1 and let $\varphi : S \to \underset{\sim}{\delta}$ be a regular Γ-norm on a set $S \subseteq \mathbb{R}$, $S \in \Gamma$ which is onto $\underset{\sim}{\delta}$. Let $G \subseteq \omega \times \mathbb{R}$ be a good universal set in $\exists^{\mathbb{R}}\Gamma$ and define

$$P_{\varphi,G} \subseteq \omega \times \underset{\sim}{\delta}$$

by

$$P_{\varphi,G}(n,\gamma) \leftrightarrow \exists x(x \in S \wedge \varphi(x) = \gamma \wedge G(n,x)).$$

Assuming AD, let $H_\Gamma = L[P_{\varphi,G}]$. The construction of H_Γ takes place inside $L(\mathbb{R})$. By Moschovakis H_Γ is the smallest inner model of ZF containing every $\exists^{\mathbb{R}}\Gamma$ subset of $\underset{\sim}{\delta}$. This led descriptive set theorists to view H_Γ as analogs of HOD at the level of Γ-definability, for Γ suitably chosen. For example, if $\Gamma = \Sigma^2_1$, then

$$H_\Gamma \cap V_{\underset{\sim}{\delta}^2_1} = \mathrm{HOD} \cap V_{\underset{\sim}{\delta}^2_1}.$$

Letting Γ be a pointclass resembling Π^1_1 and letting $\vec{\varphi}$ be a Γ-scale on a universal Γ set of reals U, Becker and Kechris have shown that the models $L[T_{2n+1}(\vec{\varphi},U)]$ are independent of the choice of scale $\vec{\varphi}$ and universal set U. As a corollary this gives that

$$L[T_\Gamma] = H_\Gamma,$$

for Γ as above. Harrington and Kechris have shown that

$$\mathbb{R} \cap L[T_{2n+1}] = C_{2n+2},$$

where

$$C_{2n+2} = \{x : x \text{ is } \Delta^1_{2n+2} \text{ in a countable ordinal}\}$$

is the largest countable Σ^1_{2n+2} set of reals.

The original intuition of Becker, Kechris and Moschovakis that H_Γ was an analog of HOD was confirmed by Steel and Woodin's HOD analysis, presented in [18], culminating in Steel's beautiful result that the H_Γ models are extender models, for Γ a pointclass resembling Π^1_1, see [18]. First using finer versions of iterability for $\mathcal{M}_{2n}$[1], Steel has shown that $\mathbb{R} \cap \mathcal{M}_{2n} = C_{2n+2}$, see [14]. In addition, for $\Gamma = \Pi^1_{2n+1}$, H_Γ is the direct limit of all countable non-dropping iterates of $\mathcal{M}_{2n}$, denoted by $\mathcal{M}_{2n,\infty}$, cutoff at the least $< \delta_\infty$-strong, which turns out in this case to be $\underset{\sim}{\delta}^1_{2n+1}$ and where δ_∞ is the least Woodin cardinal of $\mathcal{M}_{2n,\infty}$, localizing Woodin's result that $\underset{\sim}{\delta}^2_1$ is the least $< \Theta$-strong. This is essentially shown in [18]. The reader can also consult [12] for a proof that $\underset{\sim}{\delta}^2_1$ is the least $< \Theta$-strong. This also means by soundness, a strong form of acceptability, that H_Γ satisfies the GCH, a local version of Steel's result that $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies the GCH.

In [3], Hjorth has shown that under $\mathrm{Det}(\underset{\sim}{\Pi}^1_2)$, $L[T_2]$ was independent of the choice of scale on a universal Π^1_2 set of reals. For $n > 1$ the uniqueness of $L[T_{2n}]$ remained open. We show that it has a positive solution:

Theorem 1 *The models $L[T_{2n}]$ are independent of the choice of the Π^1_{2n} universal set A and of the choice of the scale $\vec{\varphi}$ on A.*

[1] The minimal proper class mouse with $2n$ Woodin cardinals.

In showing the uniqueness of $L[T_2]$, Hjorth used the Kechris-Martin theorem in an essential fashion. Jackson has shown the Kechris-Martin theorem using the theory of descriptions. The reader may consult [9] for the original proof of the Kechris-Martin theorem. The proof of the above theorem requires a generalization of this closure phenomenon to all pointclasses Π^1_{2n+1}, which we show to be true using the theory of descriptions as in the Π^1_3 case presented in [4]:

Theorem 2 ([1]) *For every $n \in \omega$, the pointclass Π^1_{2n+3} is closed under existential quantification up to κ^1_{2n+3}, where κ^1_{2n+3} is the $(2n+3)^{rd}$ Suslin cardinal of cofinality ω. In particular every Π^1_{2n+3} subset of κ^1_{2n+3} contains a Δ^1_{2n+3} member.*

After showing the uniqueness of $L[T_{2n}]$ for every $n < \omega$, we will then show the following counterpart to Steel's result on $L[T_{2n+1}]$ being an extender model:

Theorem 3 *Assume $AD^{L(\mathbb{R})}$. Let $\mathcal{M}^{\#}_{2n+1,\infty}$ be the active direct limit of all countable non-dropping iterates of $\mathcal{M}^{\#}_{2n+1}$. Then the $L[T_{2n+2}]$ are the models $L[\mathcal{M}^{\#}_{2n+1,\infty}]$ for every $n \in \omega$. In addition $L[T_{2n}]$ satisfies the GCH for every $n < \omega$.*

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2 Analysis of the model $L[T_{2n}]$

2.1 Introduction

As mentionned above, the models $L[T_{2n}]$ were not known to be independent from the universal sets and the scales the tree T_{2n} may depend on. We show now that this problem has a positive solution. Previous work of Hjorth in [3] established that $L[T_2]$ is unique.

In [2], Becker and Kechris have shown that the model $L[T_{2n+1}(P, \vec{\varphi})]$ is independent of the choice of P and $\vec{\varphi}$ on P. What Becker and Kechris actually show is a bit more: given the same assumptions as above, every Σ^1_{2n+2} (in the codes provided by the 0^{th} norm of the scale) subset of $\underset{\sim}{\delta}^1_{2n+1}$ is in the model $L[T_{2n+1}]$. We state the theorem below.

Theorem 4 (Becker, Kechris, [2]) *Let Γ be an ω-parametrized pointclass such that $\Delta^0_2 \subseteq \Gamma$, closed under recursive substitutions and under $\wedge$. Let A be a Γ-complete set of reals, let $\vec{\varphi} = \langle \varphi_n : n \in \omega \rangle$ be a regular $\exists^{\mathbb{R}}\Gamma$ scale on A and consider the 0^{th} norm $\varphi_0 : A \twoheadrightarrow \kappa$. Then for any $X \subseteq \kappa$ which is $\exists^{\mathbb{R}}\Gamma$ in the codes given by φ_0 then $X \in L[T(A, \vec{\varphi})]$*

Since every tree T_{2n+1} coming from a universal Π^1_{2n+1} set P and a regular Π^1_{2n+1} scale $\vec{\varphi}$ on P can be computed to be Σ^1_{2n+2} in the codes by the Coding lemma for every $n < \omega$, this establishes that $L[T_{2n+1}]$ is unique. Steel has shown that the $L[T_{2n+1}] = H_{2n+1}$ are extender models. Recall that H_{2n+1} is the model $L[P_{\vec{\rho},\delta}]$ where $P_{\vec{\rho},\delta}$ is a subset of $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ defined by

$$P_{\vec{\rho},\delta}(n,\alpha) \leftrightarrow \exists x(x \in P_{2n+1} \wedge \rho(x) = \alpha \wedge G(n,x)),$$

where G is a good universal set for $\exists^{\mathbb{R}}\Pi^1_{2n+1} = \Sigma^1_{2n+2}$, $\vec{\rho}$ a $\underset{\sim}{\Pi}^1_{2n+1}$ scale on P. In particular they're constructible models over a specific direct limit of a directed system of mice, see [17] and [18] for a full proof of this fact.

Our goal is to generalize Hjorth's proof that $L[T_2]$ is unique. The main difference is that we are not using the theory of sharps as in Hjorth's proof but Jackson's theory of descriptions. We first briefly recall the set up from Becker and Kechris and some previous partial results on the problem of the independence of $L[T_{2n}]$.

Definition 2.1 *Let κ^1_{2n+1} be the Suslin cardinal of cofinality ω associated to Π^1_{2n} under AD, i.e. $(\kappa^1_{2n+1})^+ = \underset{\sim}{\delta}^1_{2n+1}$.*

Let P be a complete Π^1_{2n} set of reals and let $\vec{\varphi}$ a regular Δ^1_{2n+1} scale on P. Let $\varphi_n : P \twoheadrightarrow \kappa_n$ and let $\kappa = \sup_n \kappa_n$. Then $\vec{\varphi}$ is *nice* if $\kappa = \kappa^1_{2n+1}$ and the norms φ_n satisfy the following bounded ordinal quantification condition:

$$\text{If } A(x,y) \text{ is } \Sigma^1_{2n+1} \text{ then the following is also } \Sigma^1_{2n+1}$$

$$R(n,z,x) \leftrightarrow z \in U \wedge \forall w \in U(\varphi_n(w) \leq \varphi_n(z) \rightarrow A(x,y))$$

Notice that for $n = 1$ this is essentially the Kechris-Martin theorem. With the following theorem of Becker and Kechris, the $L[T_{2n}]$ models are independent of the choice of any Π^1_{2n} complete set $A \subseteq \mathbb{R}$ and any nice scale $\vec{\varphi}$:

Theorem 5 (Becker and Kechris, [2]) *Assume AD. Let A be a complete Π^1_{2n} set of reals and let $\vec{\varphi}$ be a nice Δ^1_{2n+1} scale on A. Then the model $L[T_{A,\vec{\varphi}}]$ is independent of the choice of A and $\vec{\varphi}$.*

So basically a uniqueness of the $L[T_{2n}]$ models will require analyzing any scale by a *nice* scale. Let P be a complete Π^1_{2n} complete set of reals and let $\vec{\varphi}$ be a regular Δ^1_{2n+1} scale on P. Let κ_n be such that $\varphi_n : P \twoheadrightarrow \kappa_n$. Let then $\kappa = \sup\{\kappa_n : n \in \omega\}$. Then we have that $\kappa^1_{2n+1} \leq \kappa$. Using the scale $\vec{\varphi}$, one can define the following coding of ordinals less than κ: let

$$P^* = \{(n,x) : n \in \omega \wedge x \in P\},$$

where (n,x) denotes the new real $(n, x(0), x(1), x(2), ...)$. For $(n,x) \in P^*$, define $\varphi^*((n,x)) = \varphi_n(x)$. We will abuse the notation and drop the parenthesis around the real (n,x) when we plug in inside the norm φ^*. For κ some ordinal, we say that $X \subseteq \kappa$ is Γ in the codes provided by (P^*, φ^*) if the set

$$\{(n,x) \in P^* : \varphi^*(n,x) \in X\}$$

is in the pointclass Γ.

The above theorem relies on the following result of Becker and Kechris:

Theorem 6 (Becker, Kechris, [2]) *Assume AD. Let $X \subseteq \kappa^1_{2n+1}$ and X is Σ^1_{2n+1} in the codes provided by (P^*, φ^*). Then $X \in L[T(P, \vec{\varphi})]$, where P is a complete Π^1_{2n} set of reals and $\vec{\varphi}$ is a Δ^1_{2n+1} regular scale on P.*

To see this, let P be a complete Π^1_{2n} set of reals and let $\vec{\varphi}$ be a regular Δ^1_{2n+1} scale on P. Consider P^* as above and let ψ be the scale defined by $\psi_0(n, x) = \varphi_n(x)$ and $\psi_{k+1}(n, x) = \varphi_k(x)$. It then follows that we have that $X \in L[T(P^*, \vec{\psi})]$. We then need to see that the tree $T(P^*, \vec{\psi}) \in L[T(P, \vec{\varphi})]$. But we can compute membership in $T(P^*, \vec{\psi})$ as follows:

$$(a_0, ..., a_n), (\alpha_0, ..., \alpha_n) \in T(P^*, \vec{\psi}) \leftrightarrow \exists (b_0, ..., b_k), (\beta_0, ..., \beta_k) \in T(P, \vec{\varphi})(a_0 \leq l \wedge n + 1 \leq l \wedge a_1 = b_0 \wedge ... \wedge a_n = b_{n-1} \wedge \alpha_0 = \beta_{a_0} \wedge \forall j(k \leq j \leq n \rightarrow \alpha_j = \beta_{j-1})).$$

Throughout the proof, we will then use the 0^{th} norm ψ_0 associated to any scale $\vec{\varphi}$ as defined above and we will denote it by $\psi_{0,\vec{\varphi}}$. The goal is to show that the models $L[T_{2n}]$ are independent of the choice of an arbitrary scale not just a nice scale. We will follow Hjorth's proof, see [3] to show that an arbitrary scale can be analyzed in the model $L[T_{2n}]$ by a nice scale.

2.2 Descriptive set theory background

We recall what it means to be a regular scale:

Definition 2.2 *Let $\underset{\sim}{\Gamma} \subseteq \mathcal{P}(\mathbb{R})$ be a pointclass and let $A \in \underset{\sim}{\Gamma}$. Then a regular $\underset{\sim}{\Gamma}$-scale is a sequence $\vec{\varphi} = \langle \varphi_n : n \in \omega \rangle$ of onto maps $\varphi_n : A \twoheadrightarrow \kappa_n$, for $\kappa_n \in ORD$, satisfying the following properties:*

1. *Whenever $\{x_i\} \subseteq A$ is a sequence of reals such that $x_i \to x$ and $\varphi_n(x_i) \to \gamma_n$ for every n as $i \to \omega$, then $x \in A$ and we have the lower semi continuity property: $\varphi_n(x) \leq \gamma_n$.*
2. *The following norm relations, $\leq^*_{\varphi_n}$ and $<^*_{\varphi_n}$ are in $\underset{\sim}{\Gamma}$, for every n:*

$$x \leq^*_{\varphi_n} y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi_n(x) \leq \varphi_n(y)))$$
$$x <^*_{\varphi_n} y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi_n(x) < \varphi_n(y)))$$

Also recall that starting from a regular scale $\vec{\varphi}$, we have the tree T derived from the scale which is defined as follows

$$(s, \vec{\alpha}) \in T_{\vec{\varphi}} \longleftrightarrow \exists x (x \restriction lh(s), \varphi_0(x) = \alpha_0, ..., \varphi_{lh(s)-1}(x) = \alpha_{lh(s)-1})$$

It is then straightforward to show that $A = p[T_{\vec{\varphi}}]$ where $A \subseteq \mathbb{R}$ is the set on which the scale $\vec{\varphi}$ is. For example, if $x \in p[T_{\vec{\varphi}}]$ then use the properties of the scale to obtain $x \in A$. Notice that the tree T is on $\omega \times \kappa$ where $\kappa = \sup\{\kappa_n : n \in \omega\}$ and thus κ has to be a Suslin cardinal of cofinality ω.

We next introduce the basic ingredients of Jackson's theory of descriptions and analysis of measures in $L(\mathbb{R})$. It is not necessary for the reader to know deep facts of the theory of descriptions to follow the proof.

Recall that under determinacy successor cardinals need not be regular. As usual,

$$\underset{\sim}{\delta}^1_n =_{def} \sup\{|\preceq| : \preceq \text{ is a } \underset{\sim}{\Delta}^1_n \text{ prewellordering of } \mathbb{R}\}$$

Recall that by the coding lemma the $\underset{\sim}{\delta}^1_n$ are regular successor cardinals. By Kunen, Martin and Solovay, the $\underset{\sim}{\delta}^1_n$ are all measurable cardinals (see theorem 5.2 of [8] for a proof) and by Jackson $\underset{\sim}{\delta}^1_{2n+1}$ satisfy the strong partition property (see [4] for the underlying theory needed to prove this). We define the Suslin cardinals of cofinality ω:

$$\kappa^1_{2n+1} =_{def} \text{the least } \gamma \text{ s.t for every } A \in \underset{\sim}{\Sigma}^1_{2n+1} \text{ there exists } T \subseteq \omega \times \gamma \text{ s.t } A = p[T]$$

We have the following values for the projective ordinals and the Suslin cardinals of cofinality ω:

1. $\kappa^1_1 = \aleph_0, \underset{\sim}{\delta}^1_1 = \aleph_1$ and thus $\underset{\sim}{\delta}^1_2 = \aleph_2$,

2. $\kappa^1_3 = \aleph_\omega, \underset{\sim}{\delta}^1_3 = \aleph_{\omega+1}$ and thus $\underset{\sim}{\delta}^1_4 = \aleph_{\omega+2}$ (Martin and Solovay).

3. In general (Jackson), we have $\kappa^1_{2n+1} = \aleph_{\underbrace{\omega^{\omega^{\cdots^\omega}}}_{\text{2n+1 tower}}}, \underset{\sim}{\delta}^1_{2n+1} = \aleph_{\underbrace{\omega^{\omega^{\cdots^\omega}}}_{\text{2n+1 tower}}+1}$ and thus

   $\underset{\sim}{\delta}^1_{2n+2} = \aleph_{\underbrace{\omega^{\omega^{\cdots^\omega}}}_{\text{2n+1 tower}}+2}$

To carry out the construction of the trees T_{2n}, we need to introduce natural families of measures which arise in the context of weak and strong partition properties. We refer the reader to [4] for the notions of uniform cofinality, trees of uniform cofinality $\mathcal{R}$, the measures W^n_1 and S^n_1.

We now move towards defining $\mathrm{WO}_{\kappa^1_5}$ the set of codes of ordinals up to $\kappa^1_5 = \aleph_{\omega^{\omega^\omega}}$. Once this is done the definition of the set of codes up to $\aleph_{\epsilon_0}$ will be very similar.

Theorem 7 (Jackson, [4]) *There is a $\underset{\sim}{\Pi}^1_3$ complete set P, a Π^1_3-norm φ such that $\varphi(x) = |x| < \underset{\sim}{\delta}^1_3$ from P onto $\underset{\sim}{\delta}^1_3$ and a homogeneous tree J_3 on $\omega \times \underset{\sim}{\delta}^1_3$ for P satisfying the following. There is a c.u.b set $C \subseteq \underset{\sim}{\delta}^1_3$ such that for all $\alpha \in C$, there is a $x \in P$ with $\varphi(x) = \alpha$ and with $J_{3_x} \upharpoonright (\sup_\nu j_\nu(\alpha))$ illfounded, where the supremum ranges over measures appearing in $\mathcal{M}^{R_s}$, the tree of uniform cofinalities, coding measures which appear on a homogeneous tree projecting to WO_2.*

Next consider functions $f : \underset{\sim}{\delta}^1_3 \to \underset{\sim}{\delta}^1_3$ and the Martin tree T on $\omega \times \underset{\sim}{\delta}^1_3$. The Martin tree is the appropriate generalization of the Kunen tree. We refer the reader to [4] for the definition of the Kunen tree.

Theorem 8 (Martin, [4]) *There is a tree T on $\omega \times \underset{\sim}{\delta}^1_3$ such that for all $f : \underset{\sim}{\delta}^1_3 \to \underset{\sim}{\delta}^1_3$, there is an $x \in \mathbb{R}$ with T_x is wellfounded and a c.u.b set $C \subseteq \underset{\sim}{\delta}^1_3$ such that for all $\alpha \in C, f(\alpha) < |T_x \restriction \sup_\nu j_\nu(\alpha)|$, where if $cof(\alpha) = \omega$ then we use $|T_x \restriction \alpha|$ and if $cof(\alpha) = \omega_1$, the supremum ranges over the n-fold products, W^n_1, of the normal measure on ω_1 (these occur in the homogeneous tree construction projecting to a $\underset{\sim}{\Pi}^1_1$ set) and if $cof(\alpha) = \omega_2$, the supremum ranges over the measures occurring in the homogeneous tree construction projecting to a $\underset{\sim}{\Pi}^1_2$ set.*

Notice that the Martin tree T is Δ^1_3 in the codes. That is we can find two relations S and T which are Σ^1_3 and Π^1_3 respectively such that

$$S(n, a, x) \leftrightarrow T(n, a, x) \leftrightarrow ((a_0, ..., a_{n-1}), (|x_0|, ..., |x_{n-1}|)) \in T$$

We are now in a position to define the codes of ordinals less than κ^1_5:

Definition 2.3 (The set of codes of ordinals less than κ^1_5) *Let then T on $\omega \times \underset{\sim}{\delta}^1_3$ be the Martin tree and define*

$$WO_{\kappa^1_5} = \{\langle z, x_1, ..., x_n\rangle : z \in WO_\omega \wedge T_{x_i} \textit{ is wellfounded } \forall i\}$$

For $y = \langle z, x_1, ..., x_n\rangle \in WO_{\kappa^1_5}$, let $|y| = [f_y]_{W^n_3}$ where $f_y : (\underset{\sim}{\delta}^1_3)^n \to \underset{\sim}{\delta}^1_3$ is defined by:

$$f_y(\beta_1, ..., \beta_n) = |(T_{x_n} \restriction \sup_\nu j_\nu(\beta_n)(\delta_{n-1})|, \textit{ where },$$

$$\delta_{n-1} = |(T_{x_{n-1}} \restriction \sup_\nu j_\nu(\beta_{n-1})(\delta_{n-2})|, ...$$

$$\delta_1 = |(T_{x_1} \restriction \sup_\nu j_\nu(\beta_1)(\delta_0)|, \textit{ and } \delta_0 = |z|_{WO_\omega}$$

In the above we use the appropriate measure ν according to which cofinality the ordinal β_j has, for $1 \leq j \leq n$, in view of Martin's theorem. So for every $\alpha < \kappa^1_5, \exists y \in \mathrm{WO}_{\kappa^1_5}$ such that $\alpha = [f_y]_{W^n_3}$ for some $n \in \omega$. Notice that $\mathrm{WO}_{\kappa^1_5}$ is $\underset{\sim}{\Pi}^1_4$. Also notice that we could have defined $\mathrm{WO}_{\aleph_{\omega^{\omega^n}}}$ for each $n \in \omega$ and then taken the unions of all these sets of codes to obtain $\mathrm{WO}_{\kappa^1_5}$.

In general we define $\mathrm{WO}_{\kappa^1_{2n+3}}$ in a similar manner. Let W^n_{2n+1} the $cof(\gamma)$-cofinal measure on $\underset{\sim}{\delta}^1_{2n+1}$, where γ is the largest regular cardinal strictly less than $\underset{\sim}{\delta}^1_{2n+1}$. The Martin tree T in this case will be a tree on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ and we'll consider functions $f : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$, except this time there will be a lot more normal measures, all corresponding to the regular cardinals below $\underset{\sim}{\delta}^1_{2n+1}$.For each cofinality the appropriate measure has to be plugged in the Martin tree construction to analyze functions $f : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$.

Definition 2.4 (The set of codes of ordinals less than κ^1_{2n+3})

$$WO_{\kappa^1_{2n+3}} = \{\langle z, x_1, ..., x_m\rangle : z \in WO_{\kappa^1_{2n+1}} \wedge T_{x_i} \textit{ is wellfounded } \forall i\}$$

For $y = \langle z, x_1, ..., x_m\rangle \in WO_{\kappa^1_{2n+3}}$, let $|y| = [f_y]_{W^m_{2n+1}}$, for some $m \in \omega$, where, letting T on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ be the Martin tree, $f_y : (\underset{\sim}{\delta}^1_{2n+1})^m \to \underset{\sim}{\delta}^1_{2n+1}$ is defined by:

$$f_y(\beta_1, ..., \beta_m) = |(T_{x_m} \restriction \sup_\nu j_\nu(\beta_m)(\delta_{m-1})|, \textit{ where },$$

$$\delta_{m-1} = |(T_{x_{m-1}} \restriction \sup_\nu j_\nu(\beta_{m-1})(\delta_{m-2})|, ...$$

$$\delta_1 = |(T_{x_1} \restriction \sup_\nu j_\nu(\beta_1)(\delta_0)|, \text{ and } \delta_0 = |z|_{WO_{\kappa^1_{2n+1}}}$$

Again every ordinal below κ^1_{2n+3} is coded by a real and $\mathrm{WO}_{\kappa^1_{2n+3}}$ is a $\underset{\sim}{\Pi}^1_{2n+2}$ set of real codes. Below we state the generalized version of the Kechris-Martin theorem that we need here.

Theorem 9 *Assume $AD + V = L(\mathbb{R})$. Let X be a $\Pi^1_{2n+1}(x)$ subset of $\mathbb{R} \times \omega$. Suppose that $\exists \gamma < \kappa^1_{2n+1}$ such that for all $x \in \mathbb{R}$, for all $m \in \omega$, whenever $[f_x]_{W^m_{2n+1}} = \gamma$ then $(x, m) \in X$, for $f : (\underset{\sim}{\delta}^1_{2n-1})^m \to \underset{\sim}{\delta}^1_{2n-1}$. Then there exists a $x_0 \in \Delta^1_{2n+1}(y)$ and an $n_0 \in \omega$ such that for all $x \in \mathbb{R}$ and all $m \in \omega$, whenever $[f_x]_{W^m_{2n+1}} = [f_{x_0}]_{W^{n_0}_{2n+1}}$ then $(x, m) \in X$.*

Theorem 10 *Assume AD. Let X be a Σ^1_{2n+1} subset of $\mathbb{R} \times \mathbb{R} \times \omega$. Then the set*

$$\{x \in \mathbb{R} : \forall \gamma < \kappa^1_{2n+1} \exists y \in \mathbb{R} \exists k \in \omega([f_y]_{W^k_{2n+1}} = \gamma \wedge (x, y, k) \in X\}$$

is also Σ^1_{2n+1}.

2.3 Analyzing the trees T_{2n}

Definition 2.5 *Let Γ be a pointclass such that $\Sigma^0_1 \subseteq \Gamma$. Let $z \in \mathbb{R}$. We define the relativization $\Gamma(z)$ of Γ by: $P \subseteq \mathbb{R}$ is in $\Gamma(z)$ if there exists a set $Q \subseteq \mathbb{R}^2$ in Γ such that,*

$$P(x) \longleftrightarrow Q(z, x).$$

In particular $\Sigma^0_1(z)$ is the pointclass of semirecursive in z sets.

Definition 2.6 *Let φ be a norm on $\mathbb{R}$. We say P is invariant in x if for all $x, x' \in \mathbb{R}$ and for all $y \in \mathbb{R}$,*

$$\varphi(x) = \varphi(x') \longrightarrow [P(x, y) \leftrightarrow P(x', y)]$$

Definition 2.7 *Let $\vec{\varphi}$ be a regular scale on a set $A \subseteq \mathbb{R}$ such that $\varphi_n : A \to \kappa_n$. We say that a set $X \subseteq \mathbb{R}$ is relatively Π^1_{2n+3} invariant in the codes given by the 0^{th} norm ψ_0 if there exists a set $Y \subseteq \mathbb{R}^2$ in Π^1_{2n+3} such that*

$$x \in X \longleftrightarrow \forall x_1, ..., x_n \in A \forall k \forall i \leq n(\psi_0(k, x_i) = \alpha_{k,i} \wedge (\langle x_1, ..., x_n \rangle, x) \in Y)$$

Similarly a set $X \subseteq \mathbb{R}$ is relatively Σ^1_{2n+3} invariant in the codes given by the 0^{th} norm ψ_0 if there exists a set $Y \subseteq \mathbb{R}^2$ in Σ^1_{2n+3} such that

$$x \in X \longleftrightarrow \forall x_1, ..., x_n \in A \forall k \forall i \leq n(\psi_0(k, x_i) = \alpha_{k,i} \wedge (\langle x_1, ..., x_n \rangle, x) \in Y)$$

One can of course also let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^{n+1}$ in the above definitions.

We have the following result of Solovay, see [6],

Theorem 11 (Solovay) *Assume AD. Let $\vec{\varphi}$ be a regular Δ^1_{2n+3} scale on a a Π^1_{2n+2} set $A \subseteq \mathbb{R}$. Fix $x_1, ..., x_n \in A$. Let Λ be the pointclass of sets of reals which are relatively Π^1_{2n+3} invariant in the codes given by ψ_0. Then, $PWO(\Lambda)$.*

Recall that a pointclass Γ is ω-parametrized if there exists a $U \subseteq \omega \times \mathbb{R}$ which is universal for Γ subsets of $\mathbb{R}$.

The following is a consequence of definition 2.7 and the fact that the pointclasses Π^1_{2n+1} are all ω-parametrized for every $n < \omega$.

Lemma 2.8 *Assume AD. Let $\vec{\varphi}$ be a regular Δ^1_{2n+3} scale on a a Π^1_{2n+2} set $A \subseteq \mathbb{R}$. Fix $x_1, ..., x_n \in A$. Let Λ be the pointclass of sets of reals which are relatively Π^1_{2n+3} invariant in the codes given by ψ_0. Then Λ is ω-parametrized.*

Also we will repeatedly use in the proof the fact due to Kechris that, under Det(Γ), every prewellordering in $\exists^{\mathbb{R}}\Gamma$ does not have a perfect set of inequivalent element. (since there is no $\exists^{\mathbb{R}}\Gamma$ wellordering of $\mathbb{R}$ under Det(Γ) and since by a result of Kechris, every set in $\Game\Gamma$ has the property of Baire, see [7]). This only requires local determinacy hypothesis, although we just work under AD.

We will also use the following nice determinacy transfer result due to Kechris and Solovay, see [11]:

Theorem 12 (Kechris, Solovay) *Assume ZF+DC. Let Γ be a pointclass such that $\Delta^0_2 \subseteq \Gamma$ and Γ is a Spector pointclass. Then we have that*

$$Det(\Delta) \longrightarrow Det(\Gamma)$$

Proof. See [11] $\square$

Corollary 13 *Assume ZF+DC. Let Γ be a pointclass such that $\Delta^0_2 \subseteq \Gamma$ and Γ is a Spector pointclass. Then we have that*

$$Det(HYP) \longrightarrow Det(IND)$$

Corollary 14 *Suppose $V \models Det(\Pi^1_{2n})$. Let $\mathcal{M}$ be an inner model of ZF such that $ORD \subseteq \mathcal{M}$ and such that $\mathcal{M} \prec_{\Sigma^1_{2n+1}} V$. Then,*

$$\mathcal{M} \models Det(\Pi^1_{2n})$$

Notice that assuming Det($\utilde{\Delta}^1_{2n}$), $\mathcal{M}$ is an inner model of ZF such that ORD $\subseteq \mathcal{M}$ and such that $T_{2n+1} \in \mathcal{M}$, where T_{2n+1} is a tree on $\omega \times \utilde{\delta}^1_{2n+1}$ which projects to a universal set U and which comes from a regular Π^1_{2n+1} scale $\vec{\varphi}$ on U, we have that

$$\mathcal{M} \prec_{\Sigma^1_{2n+1}} V.$$

We will need the following theorem of Woodin. The proof follows Hjorth's argument, see [3].

Lemma 2.9 (Woodin) *Suppose $V \models Det(\Pi^1_{2n})$. Let x be a Cohen generic real over V. Then,*

$$V \prec_{\Sigma^1_{2n+2}} V[x]$$

Proof. Let T_{2n+2} be the tree coming from the Kechris-Martin scale on $\omega \times \omega \times \kappa^1_{2n+3}$ such that for some Σ^1_{2n+3} set A, $pp[T] = A$ and for some Π^1_{2n+2} set B, $p[T] = B$ and

$$A = \{x : \exists x \in \mathbb{R}((x,y) \in B)\}.$$

Let τ be a term in the forcing language for Cohen forcing. Let $\kappa^1_{2n+3} < \kappa$ be least such that $L_\kappa[T_{2n+2}, \tau]$ is admissible (i.e. satisfies KP [2]).

If x is Cohen generic over V, then $L[T_{2n+2}, \tau, x]$ is still admissible. But then by absoluteness of wellfoundedness $V[x] \models p[T_{2n+2}] \subseteq B$. Since $L_\kappa[T_{2n+2}, \tau, x]$ is admissible, if $V[x] \models \forall y((y, \tau_G(x)) \notin B)$ then for all $z \in B$ such that $(z, \tau_G(x)) \in p[T_{2n+2}]$, the fact that $(T_{2n+2})_z$ is wellfounded will be witnessed in $L_\kappa[T_{2n+2}, \tau, x]$.

But since there are only countably many reals in the model $L_\kappa[T_{2n+2}, \tau, x]$, since

$$L_\kappa[T_{2n+2}, \tau, x] \cap \mathbb{R} = Q_{2n+3}(x, z),$$

which is countable by $\mathcal{Q}$-theory[3], with τ coded by a real z, we can let x' such that $x' \in V$ and such that x' is Cohen generic over $L_\kappa[T_{2n+2}, \tau]$. Pick x' below a condition p which is such that

$$p \Vdash \text{the tree of attempts to build } y \text{ with } (y, \tau[x]) \in p[T_{2n+2}] \text{ is wellfounded}$$

Then we have that

$$L_\kappa[T_{2n+2}, \tau] \models p \Vdash \text{the tree of attempts to build } y \text{ with } (y, \tau[x]) \in p[T_{2n+2}] \text{ is wellfounded}$$

and so

$$V \models \text{the tree of attempts to build } y \text{ with } (y, \tau[x]) \in p[T_{2n+2}] \text{ is wellfounded}$$

□

Notice that since $\text{Det}(\Pi^1_{2n})$ is a Σ^1_{2n+2} statement, the above theorem can be applied to iterations of Cohen forcing, although we won't need this fact in the proof below.

Next, as in Hjorth's proof, the idea is to use forcing to analyze a given Π^1_{2n+3} norm. What the theorem below means in practice is that any tree T_{2n} which comes from a Kechris-Martin-like scale[4] can be used to analyze a given Π^1_{2n+3} norm on the reals. Call such a tree a canonical T_{2n} tree [5].

[2] KP is Kripke-Platek set theory. It is weaker than ZFC, has no power set axiom with separation and collection are limited to $\Sigma_0 (= \Delta_0 = \Pi_0)$ formulae.

[3] The fact that the set of reals of $L_\kappa[T_{2n}]$ is Q_{2n+3} can be shown using the generalizations of the Kechris-Martin theorem, this was shown in the author's thesis, see [1], section 3.4 and section 3.5.

[4] Such scales were constructed in the author's PhD thesis, jointly with Steve Jackson, this construction is not directly relevant to the proof.

[5] The canonical tree T_2 is simply the Martin-Solovay tree built using the theory of sharps.

The main twist is that once this is done, we would like to do the same for any tree T_{2n}, that is not necessarily coming from a Kechris-Martin-like scale. The corollary to the theorem below will allow making any $T_{2n} \subseteq \omega \times \kappa^1_{2n+1}$ tree look like a canonical T_{2n} tree. Once this is done, one can then capture any T_{2n}-like tree in $L[T_{2n}]$. This is the idea Hjorth used in his proof of the uniqueness of the $L[T_2]$ models.

Theorem 15 *Assume AD. Let $y \in \mathbb{R}$ and let ρ be a $\Pi^1_{2n+3}(y)$ norm on some set of reals. Let A be a complete $\Pi^1_{2n+2}(y)$ set of reals and let $\vec{\varphi}$ be a regular $\Delta^1_{2n+3}(y)$ scale. Suppose that for all $B \in \Sigma^1_{2n+3}(y)$, the following set*

$$\{x \in \mathbb{R} : \forall x_1, ..., x_n \in A, \exists y_1, ..., y_n \forall k \forall i \leq n(\psi_0(k, y_i) = \psi_0(k, x_i), (\langle y_1, ..., y_n\rangle, x) \in B)\}$$

is also $\Sigma^1_{2n+3}(y)$.

Then for every $x \in \mathbb{R}$, there exists a sequence $\{x_k\} \subseteq A$ such that for $\psi_0(k, x_i) = \alpha_i$, for every $i \leq n$ and there exists a set $D \subseteq \mathbb{R}$ which is relatively $\Delta^1_{2n+3}(y)$ invariant in the codes given by the 0^{th} norm ψ_0 satisfying the following properties:

1. *$x \in D$,*
2. *$D \subseteq \{z \in \mathbb{R} : \rho(z) = \rho(x)\}$.*

Proof. We let $y = 0$ since the case with a real parameter y is exactly the same. We will establish the theorem with a series of claims.

First we show the following claim which follows from the separation property of the pointclass of sets which are relatively Σ^1_{2n+3} invariant in the codes given by the 0^{th}-norm ψ_0.

Claim 1 *Suppose B is relatively Σ^1_{2n+3} invariant in the codes given by the 0^{th}-norm ψ_0. Suppose that*

$$\forall w, z \in B \text{ we have that } \rho(w) = \rho(z)$$

Then there exists a set B^*, such that $B \subseteq B^*$, B^* is relatively Δ^1_{2n+3} invariant in the codes given by the 0^{th}-norm ψ_0 and

$$\forall w, z \in B^* \text{ we have that } \rho(w) = \rho(z)$$

Proof.

Consider the set

$$C = \{w \in \mathbb{R} : \exists z \in B(\rho(w) \neq \rho(z))\}$$

Then the set C is relatively Σ^1_{2n+3} invariant in the codes given by ψ_0 since B is also in that pointclass. Also $C \cap B = \emptyset$. Recall that, under ZF for a nonselfdual pointclass the prewellordering property of a pointclass implies the separation property of the dual pointclass. So choose a set B^* which is relatively Δ^1_{2n+3} invariant in the codes given by ψ_0 such that $B \subseteq B^*$ and such that $C \cap B^* = \emptyset$.

□

We define the set A_0 as follows:

A_0 is the set of all $\in \mathbb{R}$ such that $\forall x_1, ..., x_n \in A$, $\forall \alpha_{k,i}$, if $\psi_0(k, x_i) = \alpha_{k,i}$, where $i \leq n$, then for every D which are relatively Δ^1_{2n+3} in ψ_0 the codes given by we have either

1. $x \notin D$, or

2. $\exists w, z \in D(\rho(w) \neq \rho(z))$

Assume that A_0 is nonempty. Then notice that $A_0 \in \Sigma^1_{2n+3}$, since $\vec{\varphi}$, and hence $\vec{\psi}$ is a Δ^1_{2n+3} scale on A, and since we can obtain, uniformly in the codes give by the 0^{th} norm ψ_0 a code for the set D, say from a universal relatively Π^1_{n+3} invariant in the codes given by ψ_0 set and since this pointclass also has the prewellordering property uniformly in the codes given by ψ_0.

Claim 2 *If $A_1 \subseteq A_0$ and $A_1 \neq \emptyset$ is relatively Σ^1_{2n+3} in the codes given by ψ_0, then $\exists w, z \in A_1$ such that $\rho(w) \neq \rho(z)$.*

Proof.
Suppose that $\forall w, z \in A_1$, we have that $\rho(w) = \rho(z)$, then let $A_1 \subseteq A_2$ such that A_2 is relatively Δ^1_{2n+3} in the codes given by ψ_0 and $\forall w, z \in A_2$, we have $\rho(w) = \rho(z)$. But now notice that $A_2 \cap A_0 = \emptyset$, by definition of A_0 and then we must have $A_1 = \emptyset$. Contradiction!

$\square$

Now we define the following partial order $\mathbb{P}$:

$$\mathbb{P} = \{B \subseteq \mathbb{R} : B \neq \emptyset, B \subseteq A_0, \exists \{x_i\}_{i \leq n} \subseteq A \psi_0(k, x_i) = \alpha_{k,i} \text{ and } B \text{ is rel. } \Sigma^1_{2n+3} \text{ inv. in } \psi_0\}$$

For $B_0, B_1 \in \mathbb{P}$, we let

$$B_0 \leq_{\mathbb{P}} B_1 \longleftrightarrow B_0 \subseteq B_1.$$

Notice that by assumption $\mathbb{P} \neq \emptyset$.

Let V_λ a large enough rank initial segment of V such that $V_\lambda \vDash \mathrm{ZFC}^-$. Let $X \prec V_\lambda$ be a countable elementary substructure of V_λ and let M be the transitive collapse of X. Let $\mathbb{Q} = \mathbb{P} \cap M$ and let $\leq_{\mathbb{Q}} = \leq_{\mathbb{P}} \cap \mathbb{Q} \times \mathbb{Q}$.

If G is $\mathbb{Q}$-generic over V, we let x_G be the real introduced by forcing with $\mathbb{Q}$. We also let $\dot{G}$ be a name for the $\mathbb{Q}$ generic G.

Claim 3 $(A_0, A_0) \Vdash \rho(x_{\dot{G}_0}) \neq \rho(x_{\dot{G}_1})$.

Proof.
Suppose that there are conditions $B_0 \subseteq A_0$ and $B_1 \subseteq A_0$ such that

$$(B_0, B_1) \Vdash \rho(x_{\dot{G}_0}) = \rho(x_{\dot{G}_1})$$

Let

$$B_0^* = B_0 \times B_0 \cap \{(w, z) : \rho(w) \neq \rho(z)\}$$

Then since $\mathbb{Q}$ is countable, we have by elementarity of M that $B_0^* \in M$. Also $B_0^* \neq \emptyset$ by the above claim. Let

$$\mathbb{Q}' = \{B \subseteq \mathbb{R}^2 : B \in M, B \neq \emptyset, B \text{ is rel. } \Sigma^1_{2n+3} \text{ inv. in the codes } \alpha_{k,i} \text{ given by } \psi_0(k, x_i)\}$$

Let (K, G) be $\mathbb{Q}' \times \mathbb{Q}$ generic over V such that $K \subseteq B_0^* \wedge G \subseteq B_1$. Let

$$G^0 = \{B_0 \subseteq \mathbb{R} : \{(w, z) \in B_0^* : z \in B_0\} \in H\}$$

and let

$$G^1 = \{B_1 \subseteq \mathbb{R} : \{(w, z) \in B_1^* : z \in B_0\} \in H\}$$

Notice that (G^0, G) and (G^1, G) are both $\mathbb{P} \times \mathbb{P}$ generic over V [6]. Also since $B_0 \in G^0$, $B_0 \in G^1$ and $B_1 \in G$ we have that

$$\rho(x_{G^0}) = \rho(x_G) \text{ and } \rho(x_{G^1}) = \rho(x_G)$$

Since A is a complete Π^1_{2n} set, any Π^1_{2n} set $X \subseteq \mathbb{R}^2$ which projects to $(\leq^*_\rho)^c$ is such that $X \leq_W A$. Let ε be a real coding the function Wadge reducing X to A. Then this fact continues to hold in $V[H, G]$ with $\varepsilon \in V[H, G]$. In addition, by absoluteness of wellfoundedness we have that $V[H, G] \vDash p[T_{2n+2}] \subseteq A$. Let $\bar{\varepsilon} = \pi^{-1}(\varepsilon)$, so that $\bar{\varepsilon}$ codes the Wadge reduction inside M. Since π naturally lifts to generic extensions. By genericity of G^0, G^1, we then have reals x_{G^0} and x_{G^1} such that

$$\rho(x_{G^0}) \neq \rho(x_{G^1}).$$

But then $\rho(x_{G^0}) = \psi(x_G)$ and $\rho(x_{G^1}) = \rho(x_G)$ yet $\rho(x_{G^0}) \neq \rho(x_{G^1})$ in $V[H, G]$. Since $\mathbb{Q} \times \mathbb{P}$ is countable then $V[H, G]$ is equivalent to $V[x]$ for x a Cohen real. Contradiction!

□

To finish the proof of the theorem, we use the following basic lemma from forcing theory:

Lemma 2.10 *Let z be a Cohen real. Then there is a perfect set F in $V[x]$ such that for every $F' \subseteq F$, $F' = \{z_0, ..., z_j\}$ finite, we have z_j is generic over $V[z_0, ..., z_{j-1}]$.*

Proof.
Consider the following poset:

$$\mathbb{P} = \{(T, k) : T \subseteq 2^{<\omega}, ht(T) = k\}$$

We also let

$$(T, k) \leq (S, l) \longleftrightarrow S \subseteq T \wedge l \leq k.$$

Any $\mathbb{P}$-generic/V adds a perfect tree U. Let G be $\mathbb{P}$-generic over V. Let $z_0, ..., z_j \in U$ be in $V[G]$. Let $(T, k) \in V$ such that for branches $f_0, ..., f_j \in [T]$ we have $f_0 \subseteq z_0, ..., f_j \subseteq z_j$. Notice that there are densely many conditions $(S, l) \leq (T, k)$ for which there exists a conditions (R, m) such that for branches $f_0^0, ..., f_j^0 \in [R]$ we have $f_0 \subseteq f_0^0, ..., f_j^0 \subseteq f_j$ and $N_{f_0^0} \times ... \times N_{f_j^0} \cap X = \emptyset$ for some nowhere dense set X. But since G is generic, it has one such condition. So $(z_0, ..., z_j) \notin X$,and it is a sequence of Cohen reals, so z_j is generic over $V[z_0, ..., z_{j-1}]$.

□

So let z be a Cohen real and let F be a perfect set, in $V[z]$, of $\mathbb{R}$-many Cohen reals x_f, $f \in 2^\omega$ such that if $f \neq g$ there exists G_f and G_g satisfying the following:

[6] One can use a genericity argument to show this.

1. (G_f, G_g) are mutually V-generic below (A_0, A_0) for $\mathbb{P} \times \mathbb{P}$

2. $x_{G_f} = f, x_{G_g} = g$ and $\rho(x_f) \neq \rho(x_g)$.

But F is in V, since the second clause above is Σ^1_{2n+2} and since $V \prec_{\Sigma^1_{2n+2}} V[z]$. But ρ was supposed to be a Π^1_{2n+3} norm. Contradiction!

$\square$

Corollary 16 *Assume AD. Let ρ be a $\Pi^1_{2n+3}(y)$ norm on some set of reals. Then $\forall x \in \mathbb{R}, \exists \{\alpha_k\} \subseteq (\kappa^1_{2n+3})^{<\omega}, \exists D$ which is relatively Δ^1_{2n+3} in the codes given by some scale $\vec{\varrho}$ such that*

1. $x \in D$

2. $D \subseteq \{z \in \mathbb{R} : \rho(z) = \rho(x)\}$.

Proof.

Since we don't have the assumption on the norms of the scale $\vec{\varrho}$ as in the above theorem, we use the Kechris-Martin theorem. Then the set A_0 defined in the above claims is Σ^1_{2n+3} by the Kechris-Martin theorem. If $f_x : (\underset{\sim}{\delta}^1_{2n+1})^k \to \underset{\sim}{\delta}^1_{2n+1}$ and $f_y : (\underset{\sim}{\delta}^1_{2n+1})^j \to \underset{\sim}{\delta}^1_{2n+1}$ are two functions coded by the "nesting" defined for generalized Martin tree, and if $[f_x]_{W^k_{2n+1}} = [f_y]_{W^j_{2n+1}}$ and if $\psi_{0,\varrho_k}(x) = \alpha_{0,k}, \psi_{0,\varrho_j}(x) = \beta_{0,j}$ then the pointclass of relatively Δ^1_{2n+3} invariant in the codes given by ψ_{0,ϱ_k} for some $\{x_i\}_{i \leq k}$ and the pointclass of relatively Δ^1_{2n+3} invariant in the codes given by ψ_{0,ϱ_j} for some $\{x_i\}_{i \leq j}$ are the same. So one can always find new codes in ψ_0 for some sequence of real such that the corollary holds.

$\square$

Corollary 17 *Assume AD. Let ρ be a $\Pi^1_{2n+3}(y)$ norm on some set of reals. Then $\forall x \in \mathbb{R}, \exists j \in \omega, \exists \alpha < \kappa^1_{2n+3}$ such that there exists a $D \subseteq \mathbb{R}$ such that*

1. $\exists y \in \mathbb{R}([f_y]_{W^j_{2n+3}} = \alpha)$

2. $\forall y \in \mathbb{R}([f_y]_{W^j_{2n+3}} = \alpha \longrightarrow D$ is invariantly $\Delta^1_{2n+3}(y))$

3. $x \in D$

4. $D \subseteq \{z \in \mathbb{R} : \rho(z) = \rho(x)\}$.

So basically D is Δ^1_{2n+3} in the equivalence classes functions $f : (\underset{\sim}{\delta}^1_{2n+1})^{<\omega} \to \underset{\sim}{\delta}^1_{2n+1}$

2.4 The Main Theorem on the uniqueness of $L[T_{2n}]$

We assume AD again throughout this section. We start with the following basic lemma from $\mathcal{Q}$-theory:

Lemma 2.11 ([10]) *Assume AD. Then there exists a non trivial Π^1_{2n+3} singleton, i.e. a $y_{2n+3} \in \mathbb{R}$ such that $\{y_{2n+3}\} \in \Pi^1_{2n+3}$ and $y_{2n+3} \notin \Delta^1_{2n+3}$.*

Next, we aim to see that any Π^1_{2n+3} subset of κ^1_{2n+3} is uniformly $\Delta^1_{2n+3}(y_{2n+3})$.

Lemma 2.12 *Assume AD. Let $A \subseteq \mathbb{R}^2$ be a universal Π^1_{2n+3} set (recall that Π^1_{2n+3} is ω-parametrized). Suppose that $\{y_{2n+3}\} = A_t$, for some $t \in \omega$, and $y_{2n+3} \notin \Delta^1_{2n+3}$. Suppose ψ is a Π^1_{2n+3} norm on the set A.*

Then $\forall \alpha < \kappa^1_{2n+3}, \forall k, l \in \omega$, we have

$$\forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \rightarrow A(k,w)) \leftrightarrow \exists z \in \mathbb{R}, \exists j \in \omega[[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3}),$$

where $\boldsymbol{d} : (\omega)^3 \rightarrow \omega$ is a recursive function such that for all $z \in \mathbb{R}$ and for all $k, j, l \in \omega$,

$$A(\boldsymbol{d}(k,j,l)),z)) \leftrightarrow \forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = [f_z]_{W^j_{2n+1}} \rightarrow A(k,w))$$

Proof.

Notice that our hypothesis on $\boldsymbol{d}$ immediately gives that

$$\exists z \in \mathbb{R}, \exists j \in \omega[[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3}) \longrightarrow \forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \longrightarrow A(k,w))$$

Suppose the conclusion of the lemma fails. Then there must be $l \in \omega$ and $\alpha < \kappa^1_{2n+3}$ such that for all $z \in \mathbb{R}$, $\forall j \in \omega$, whenever we have that $[f_z]_{W^j_{2n+1}} = \alpha$ then

$$A(\boldsymbol{d}(k,j,l),z)) \wedge \psi(t, y_{2n+3}) \leq \psi((\boldsymbol{d}(k,j,l),z))$$

But now this implies that

$$\{y_{2n+3}\} \in \Delta^1_{2n+3}(z),$$

by assumption. This then gives that

$$y_{2n+3} \in \Delta^1_{2n+3}(z)$$

and

$$\forall z \in \mathbb{R}, \forall j \in \omega([f_z]_{W^j_{2n+1}} = \alpha \longrightarrow \exists y \in \Delta^1_{2n+3}(z)(A(t,y)))$$

By notice that by restricted quantification, we have that

$$B(z) \longleftrightarrow \exists y \in \Delta^1_{2n+3}(z)(A(t,y))$$

is also Π^1_{2n+3} and by Kechris-Martin we have

$$\exists x \in \Delta^1_{2n+3} \text{ such that } \exists y \in \Delta^1_{2n+3}(x)(A(t,y))$$

and hence

$$\exists y \in \Delta^1_{2n+3}(A(t,y))$$

Contradiction!

$\square$

Lemma 2.13 *Assume AD. Let A be a universal Π^1_{2n+3} set of reals and let $\boldsymbol{d}$ be as above. Let $M \prec_{\Sigma^1_{2n+3}} V$ be a transitive inner model of ZF+DC such that ORD $\subseteq M$. Then $\exists y \in M \cap \mathbb{R}, \exists t \in \omega$ such that $A(t,y)$ and for all $\alpha < \kappa^1_{2n+3}$, for all $k,l \in \omega$, we have that*

$$\forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \rightarrow A(k,w)) \leftrightarrow \exists z \in \mathbb{R}, \exists j \in \omega [f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3})$$

Proof. By assumption, M satisfies Π^1_{2n+2}-determinacy. So

$$M \vDash \forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \rightarrow A(k,w)) \leftrightarrow \exists z \in \mathbb{R}, \exists j \in \omega [f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3})$$

Also by assumption and since $M \vDash$ "$A(k,w)$ holds" then we have that $A(k,w)$ really holds. So have that

$$\exists z \in \mathbb{R}, \exists j \in \omega [f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3})$$

implies that

$$\forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \rightarrow A(k,w))$$

Now suppose that there is an $l \in \omega, \exists \alpha < \kappa^1_{2n+3}$ such that $\forall z \in \mathbb{R} \forall j \in \omega$ whenever $[f_z]_{W^j_{2n+1}} = \alpha$ then we have that $\psi(t, y_{2n+3}) \leq \psi((\boldsymbol{d}(k,j,l),z))$. Since this is a $\Pi^1_{2n+3}(y_{2n+3})$ statement about α, by Kechris-Martin $\exists x \in \Delta^1_{2n+3}(y_{2n+3})$ and $t \in \omega$ such that $[f_x]_{W^t_{2n+1}} = \alpha$. But then x is definable in M thus $x \in M$. Since $M \vDash \psi(\boldsymbol{d}(k,t,j),x) < \psi(k,w)$ by assumption. But we have $M \prec_{\Sigma^1_{2n+3}} V$. Contradiction!

$\square$

Finally in the next last two lemmas we use the fact that every Π^1_{2n+3} subset of κ^1_{2n+3} is uniformly $\Delta^1_{2n+3}(y_{2n+3})$ to compute any Δ^1_{2n+3} scale $\vec{\varrho}$ in a nice scale $\vec{\varphi}$.

Lemma 2.14 *Assume AD. Let P and Q be two universal $\Pi^1_{2n+2}(y_{2n+3})$ sets of reals. Let $\vec{\varphi}$ be a $\Delta^1_{2n+3}(y_{2n+3})$ scale on P and $\vec{\rho}$ a $\Delta^1_{2n+3}(y_{2n+3})$ scale on Q. Consider the trees from the scales $T_{2n+2}(P,\vec{\varphi})$ and $T_{2n+2}(Q,\vec{\rho})$. Suppose that for every $B \in \Sigma^1_{2n+3}(y_{2n+3})$, the following set*

$$\{x \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall k \leq n, (\langle y_1, ..., y_n \rangle, x) \in B)\}$$

is also $\Sigma^1_{2n+3}(y_{2n+3})$. Then $T_{2n+2}(Q,\vec{\rho}) \in L[T(\vec{\varphi}), y_{2n+3}]$.

Proof.

Since we're assuming AD, all relevant pointclass are ω-parametrized, in particular, the pointclass of sets which are relatively Σ^1_{2n+3} invariantly in the codes is ω-parametrized uniformly in the codes given by $\psi_{0,\vec{\varphi}}$. So we can find a set $U \subseteq \omega \times \mathbb{R} \times \mathbb{R}$ which is $\Pi^1_{2n+3}(y_{2n+3})$ and such that

1. $\forall x_1, ..., x_n, \forall w_1, ..., w_n \in P, \forall k \in \omega, \forall l \forall i \leq n$
   $(\psi_{0,\vec{\varphi}}(l, x_i) = \psi_{0,\vec{\varphi}}(l, w_i) \longrightarrow \{x \in \mathbb{R} : (x, \langle x_i \rangle, k) \in U\} = \{x \in \mathbb{R} : (x, \langle w_i \rangle, k) \in U\}$
2. $\forall x_1, ..., x_n \in P$ whenever $\psi_{0,\vec{\varphi}}(l, x_i) = \kappa_{l,i}$ and W is relatively Π^1_{2n+3} invariant in the codes $\kappa_{0,0}, ..., \kappa_{l,i}$, then $\exists k \in \omega$ such that $W = \{x \in \mathbb{R} : (x, \langle x_i \rangle, k) \in U\}$

Let $\vec{\kappa}$ denote the sequence of ordinals $\kappa_{0,0}, ..., \kappa_{l,i}$. Now let $U_{\vec{\kappa},k}$ denote projection of U onto the first coordinate, i.e. the set

$$\{x \in \mathbb{R} : (x, \langle x_i \rangle, k) \in U\}.$$

Next consider the set

$$\mathcal{U}_n = \{(\vec{\kappa}, k) : U_{\vec{\kappa},k} \text{ is rel. } \Delta^1_{2n+3} \text{ inv. } , U_{\vec{\kappa},k} \neq \emptyset, \forall x, y \in U_{\vec{\kappa},k}(\psi_{0,\vec{\rho}}(l, x_0) = \psi_{0,\vec{\rho}}(l, y_0), \forall l \leq n)\}$$

This is basically the set of codes of sections of relatively Δ^1_{2n+3} in the codes sets of reals but we just require that they're invariant in the norm being analyzed by the Kechris-Martin norm. Also we have that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$. For any $(\vec{\kappa}, k)$ and $(\vec{\gamma}, j)$, we define $(\vec{\kappa}, k) \leq_n (\vec{\gamma}, j)$ if and only if for every $x \in U_{\vec{\kappa},k}$ and for every $y \in U_{\vec{\gamma},j}$,

$$\psi_{0,\vec{\rho}}(n, x) \leq \psi_{0,\vec{\rho}}(n, y).$$

But by Becker and Kechris, we have that $(\mathcal{U}_n, \leq_n)$ is in $L[T(\vec{\varphi}), y_{2n+3}]$ since the prewellordering $\leq_n$ is $\Sigma^1_{2n+3}(y_{2n+3})$ in the codes and since that sets $\mathcal{U}_n$ are also $\Sigma^1_{2n+3}(y_{2n+3})$ in the codes. By theorem 17, we can also find a code $(\vec{\kappa}, k) \in \mathcal{U}_n$ for every $n \in \omega$, for every $x \in Q$, $x \in U_{\vec{\kappa},k}$, since these are exactly the codes of relatively Δ^1_{2n+3} in the codes sets of reals. Next for each $n \in \omega$, let $\varrho_n : \mathcal{U}_n \to \zeta_n$ be the norm associated to the prewellordering $\leq_n$ defined above:

$$\text{for any codes } (\vec{\kappa}, k) \text{ and } (\vec{\gamma}, j) \text{ in } \mathcal{U}_n, \varrho_n((\vec{\kappa}, k)) < \varrho_n((\vec{\gamma}, j)) \text{ iff } (\vec{\kappa}, k) <_k (\vec{\gamma}, j)$$

Notice that for every $n \in \omega$, $\zeta_n < \kappa^1_{2n+3}$. By Becker and Kechris, the sequence of norms $\vec{\varrho}$ is in $L[T(\vec{\varphi}), y_{2n+3}]$. Since $T(\vec{\rho})$ is the set

$$\{\vec{\alpha} \in \text{ORD}^{<\omega} : \exists n \in \omega, lh(\vec{\alpha}) = n, \exists (\vec{\kappa}, k) \in \mathcal{U}_n \text{ such that } \forall l \leq n, \varrho_n((\vec{\kappa}, k)) = u(n)\}$$

then $T(\vec{\rho}) \in L[T(\vec{\varphi}), y_{2n+3}]$ and we are done.

□

We finally conclude with the last lemma which finishes the proof that the models $L[T_{2n+2}]$ are unique.

Lemma 2.15 *Assume AD. Let P and Q be two universal Π^1_{2n+2} set of reals. Let $\vec{\varphi}$ be a Δ^1_{2n+3} scale on P and $\vec{\rho}$ be a Δ^1_{2n+3} scale on Q. Consider the trees from the scales $T(\vec{\varphi}) = T_{2n+2}(P, \vec{\varphi})$ and $T(\vec{\rho}) = T_{2n+2}(Q, \vec{\rho})$ as usual. Then $L[T(\vec{\varphi})] = L[T(\vec{\rho})]$*

Proof.

By the previous lemma, we just have to show that $T(\vec{\rho}) \in L[T(\vec{\varphi})]$. By lemma 2.21, we only need to see that if $y \in \mathbb{R}$ is such that for $L[T(\vec{\varphi})] \prec_{\Sigma^1_{2n+3}} V$, $y \in L[T(\vec{\varphi})] \cap \mathbb{R}$ and satisfies the conclusion of lemma 2.20, then for all sets B which are $\Sigma^1_{2n+3}(y)$, then

$$\{x \in \mathbb{R} : \forall x_1, ..., x_n(x_i \in P \to \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall i \leq n, \forall k, (\langle y_1, ..., y_n \rangle, x) \in B)\}$$

is also $\Sigma^1_{2n+3}(y)$. By the proof we give in the next section of the fact that $L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$, y can be considered to be y^0_{2n+3}, the least non-trivial Π^1_{2n+3} singleton.

Next we define a Π^1_{2n+3} norm Φ for which the above lemma applies, by setting $\Phi(x) = \Phi(y)$ if and only if either

1. $x = \langle x_i \rangle, y = \langle y_i \rangle, \forall i \leq n$, for some $n \in \omega$, and $\forall i \leq n, x_i \in P \wedge y_i \in P \wedge \psi_{0,\vec{\varphi}}(k, x_i) = \psi_{0,\vec{\varphi}}(k, y_i)$, or

2. $x \neq \langle x_i \rangle$ and either for every $i \leq n$, $x_i \notin P$ or there exists an $i \in \omega$ such that $x_i \notin P$ and $y \neq \langle y_i \rangle$ and either $i \leq n$, $y_i \notin P$ or there exists an $i \in \omega$ such that $y_i \notin P$

Next we fix a set $U \subseteq \mathbb{R} \times \mathbb{R} \times \omega$ such that

1. For all $j, l \in \omega$ for all $w, z \in \mathbb{R}$ and for all $t \in \omega$
$$[f_x]_{W^l_{2n+1}} = [f_y]_{W^j_{2n+1}} \to \{z : A(z, l^\frown x, t)\} = \{z : A(z, j^\frown y, t)\},$$

2. $U \in \Pi^1_{2n+3}$,

3. For every $\alpha < \kappa^1_{2n+3}$, whenever $W = \{z : \forall x([f_x]_{W^l_{2n+1}} = \alpha \to V(z, x)\}$ where $V \in \Pi^1_{2n+3}$, then there is $t \in \omega$, $y \in \mathbb{R}$ and $j \in \omega$ such that $W = \{z : U(z, j^\frown y, t)\}$.

For $t \in \omega$, $\alpha < \kappa^1_{2n+3}$ and $[f_x]_{W^l_{2n+1}} = \alpha$ we consider as in lemma 2.20, the projection of U onto the first coordinate:
$$U_{\alpha,t} = \{z \in \mathbb{R} : U(z, l^\frown x, t)\}.$$
By lemma 2.20, the assumption on y^0_{2n+3} implies that for $B \in \Pi^1_{2n+3}$, we have that
$$\{(x, l) : \forall (y, j) \in \mathbb{R} \times \omega([f_x]_{W^l_{2n+1}} = [f_y]_{W^j_{2n+1}} \to B(y, j))\}$$
is $\Sigma^1_{2n+3}(y^0_{2n+3})$. We now fix a set $B \in \Sigma^1_{2n+3}(y^0_{2n+3})$.

Let X be the set of all $z \in \mathbb{R}$ such that for all $\alpha < \kappa^1_{2n+3}$ and for all t_1:

1. Either for all $t_2 \in \omega, U_{\alpha,t_1} \neq_\Phi U_{\alpha,t_2}$, or

2. there are $x, y \in U_{\alpha,t_2}$ which are not Φ-equivalent, or
3. $U_{\alpha,t_2} = \emptyset$, or
4. There exists an $x \in U_{\alpha,t_2}$ such that $x = \langle x_i \rangle, \forall i < \omega, x_i \in P \wedge \exists y = \langle y_i \rangle$ such that $\Phi(x) = \Phi(y)$ and $B(y, z)$, or
5. There is an $x \in U_{\alpha,t_2}$ such that either $x \neq \langle x_i \rangle$ for all x_i or $x = \langle x_i \rangle$ and for some $i \in \omega$, $x_i \notin P$.

Claim 4 *X is $\Sigma^1_{2n+3}(y^0_{2n+3})$*

Proof.
We check that the clauses (1) through (5) are at most $\Sigma^1_{2n+3}(y^0_{2n+3})$. Clause (1) is $\Sigma^1_{2n+3}(y^0_{2n+3})$ since the pointclass Π^1_{2n+3} has the prewellordering property. Taking the existential quantifier in clause (2) outside the conjunction of clauses (1) and (2), shows that (1)$\vee$(2) is also $\Sigma^1_{2n+3}(y^0_{2n+3})$. The same holds for (1)$\vee$(3), (1)$\vee$(4) and (1)$\vee$(5). By the generalization of the Kechris-Martin theorem, X is now $\Sigma^1_{2n+3}(y^0_{2n+3})$.

$\square$

This last claim now finishes the proof of the lemma:

Claim 5 *We have that*

$$X = \{z \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall k \forall i \leq n, (\langle y_1, ..., y_n \rangle, x) \in B\}$$

Proof. Let $x_1, ..., x_n \in P_0$ and let $\psi_{0,\vec{\varphi}}(k, x_i) = \alpha_{k,i}$ for all $k \in \omega$ and $i \leq n$ then by corollary 4.24, there exists $\alpha < \kappa^1_{2n+1}$ and $t_2 \in \omega$ such that

1. U_{α,t_2} is Δ^1_{2n+3} in any code w which codes a function $f : (\underset{\sim}{\delta}^1_{2n+1})^{<\omega} \to \underset{\sim}{\delta}^1_{2n+1}$ via the "nesting" of the Martin tree and which equivalence class gives α and
2. $x = \langle x_i \rangle \in U_{\alpha,t_2}$ and
3. For every $y \in U_{\alpha,t_2}$, we have $y = \langle y_i \rangle$ with $\psi_{0,\vec{\varphi}}(k, y_i) = \alpha_{k,i}$, and so we have $\Phi(y) = \Phi(x)$.

Hence if the defining condition of the set

$$\{z \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i)), \forall k \leq n, (\langle y_1, ..., y_n \rangle, x) \in B\}$$

fails, then U_{α,t_2} witnesses that $z \in \mathbb{R} \notin X$. Conversely, if $z \notin X$ then clause (4) above must fail and thus

$$z \notin \{z \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall k \leq n, (\langle y_1, ..., y_n \rangle, x) \in B\}.$$

$\square$

$\square$

This completes the proof of the main theorem.

3 $L[T_{2n}]$ and direct limits associated to mice

3.1 Introduction

In this section the goal is to show that $L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$. We'll use ideas of Sargsyan and Steel to show the main theorem below.

The following theorem is a central theorem in descriptive inner model theory. It jump-started the analysis of HOD's of models of determinacy.

Theorem 18 (Steel [16])) *$AD^{L(\mathbb{R}}$ implies that $HOD^{L(\mathbb{R})}$ is a core model below Θ. In $L(\mathbb{R})$ every regular cardinal below Θ is measurable.*

We will show the following theorem below:

Theorem 19 (Main Theorem) *Assume $AD^{L(\mathbb{R})}$. Then the $L[T_{2n+2}]$ are the models $L[\mathcal{M}^{\#}_{2n+1,\infty}]$.*

We need to record all the notions involved in the computation. Given a set of reals A, $\Game A$ is defined as follows:

$$x \in \Game A \leftrightarrow \exists n_0 \forall n_1 \exists n_2 \forall n_3 ... (x, \{(i, n_i) : i \in \omega\}) \in A$$

Notice that this is the same as saying :

$$\Game A = \{x : \text{I has a winning strategy in } G_{A_x}\}$$

Let $\mathcal{M}$ be a premouse. For $\alpha < o(\mathcal{M})$, we let $\mathcal{M}||\alpha$ be $\mathcal{M}$ cutoff at α and the last predicate indexed at α is kept. $\mathcal{M}|\alpha$ is $\mathcal{M}||\alpha$ without its last predicate. We say that α is a cutpoint if there are no extenders on the extender sequence of $\mathcal{M}$ such that $\alpha \in (cp(E), lh(E)]$. We say α is a strong cutpoint is there are no extender on the extender sequence of $\mathcal{M}$ such that $\alpha \in [cp(E), lh(E)]$. We refer the reader to [17] for the definitions of the iteration game and normal trees.

Definition 3.1 *Let $\mathcal{T}$ be an n-normal iteration tree of limit length on an n-sound premouse $\mathcal{M}$ and let b be a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b, \mathcal{T})$ is the shortest initial segment $\mathcal{Q}$ of $\mathcal{M}^{\mathcal{T}}_b$, if one exists, such that $\mathcal{Q}$ projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ if not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.*

We refer the reader to [17] for the definition of the Dodd-Jensen property. The property says that iteration maps are minimal. The main use of the Dodd-Jensen property is in showing that HOD limits exist.

Definition 3.2 *(C_Γ) For a a countable transitive set we let*

$$C_\Gamma(a) = \{b \subseteq a : b \in OD(a)\} = \mathcal{P}(a) \cap Lp^\Gamma(a)$$

where $Lp^\Gamma(a)$ is the union of all a premice projecting to a having an ω_1 iteration strategy in Γ.

let Γ_n be such that $C_{\Gamma_n}(x) = \mathbb{R}^{\mathcal{M}_n(x)}$. So we'll let Γ_ω be $(\Sigma^2_1)^{L(\mathbb{R})}$.

Definition 3.3 *Let Γ_n be as above. $\mathcal{N}$ is called Γ_n-suitable if there is a δ such that $\mathcal{N} = Lp^{\Gamma_n}(\mathcal{N} \mid \delta)$ and*

1. *$\mathcal{N} \vDash \delta$ is Woodin*
2. *For every $\eta < \delta$,*
   (a) *If η is a cutpoint of $\mathcal{N}$ then $Lp^{\Gamma_n}(\mathcal{N} \mid \eta) \trianglelefteq \mathcal{N}$*
   (b) *$Lp^{\Gamma_n}(\mathcal{N} \mid \eta) \vDash \eta$ is not Woodin, and*
   (c) *If η is a strong cutpoint of $\mathcal{N}$, then $Lp^{\Gamma_n}(\mathcal{N} \mid \eta) = \mathcal{N} \mid (\eta^+)^{\mathcal{N}}$*

We write $\delta^{\mathcal{N}}$ for the unique such δ.

A theorem of Steel and Woodin states that a real x is ordinal definable from y over $L(\mathbb{R})$ if and only if x is in $\mathcal{M}_\omega(y)$, where $\mathcal{M}_\omega(y)$ is the minimal class premouse over y with ω many Woodin cardinals, see chapter 7 of [17] for a proof of this theorem. This phenomenon is called mouse capturing. By mouse capturing in $L(\mathbb{R})$, $(\Sigma^2_1)^{L(\mathbb{R})}$-suitability is the same as $L(\mathbb{R})$-suitability, in which case we just say "suitable". Given an iteration tree $\mathcal{T}$ on a suitable mouse $\mathcal{N}$, $\mathcal{T}$ is correctly guided if for every limit $\alpha < \mathrm{lh}(\mathcal{T})$, if b if the branch of $\mathcal{T} \upharpoonright \alpha$ chosen by $\mathcal{T}$ and $Q(b, \mathcal{T} \upharpoonright \alpha)$ exists then

$$Q(b, \mathcal{T} \upharpoonright \alpha) \trianglelefteq Lp(\mathcal{N}(\mathcal{T} \upharpoonright \alpha).$$

$\mathcal{T}$ is said to be short if either $\mathcal{T}$ has a last model or there is a wellfounded branch b such that $\mathcal{T}^\frown\{\mathcal{N}^{\mathcal{T}}_b\}$ is correctly guided. $\mathcal{T}$ is maximal if $\mathcal{T}$ is not short. Notice that maximal trees can't be normally continued since every initial segment of a normal tree is short.

Definition 3.4 *Let $\mathcal{N}$ be suitable. then $\mathcal{N}$ is short tree iterable iff whenever $\mathcal{T}$ is a short tree on $\mathcal{N}$ then:*

1. *If $\mathcal{T}$ has a last model then it can be freely extended by one more ultrapower, that is every putative normal tree $\mathcal{U}$ extending $\mathcal{T}$ and having length $\mathrm{lh}(\mathcal{T}) + 1$ has a wellfounded last model, and*
2. *If $\mathcal{T}$ has limit length and $\mathcal{T}$ is short , then $\mathcal{T}$ has a cofinal wellfounded branch.*

As usual for a suitable mouse $\mathcal{N}$ we let

$$\gamma^{\mathcal{N}}_s = \sup(Hull^{\mathcal{N}}(s^-) \cap \delta^{\mathcal{N}}),$$

$$\mathrm{Th}^{\mathcal{N}}_s = \{(\varphi, t) : t \in (\delta^{\mathcal{N}} \cup s^-)^{<\omega} \wedge L[\mathcal{N} | \max(s)] \vDash \varphi(t)\},$$

and

$$H^{\mathcal{N}}_s = Hull^{\mathcal{N}}(\gamma^{\mathcal{N}}_s \cup \delta^{\mathcal{N}})$$

We say $\mathcal{N}$ is n-iterable if whenever $\mathcal{T}$ is a normal tree on $\mathcal{N}$ there is a correct branch b of $\mathcal{T}$ such that $i_b(s_n) = s_n$, where s_n is the sequence of the first n uniform indiscernibles, then $i_b \restriction H^{\mathcal{N}}_{s_n}$ is independent of the branch b. We let $i^n_{\mathcal{N},\mathcal{Q}}$ be the iteration embedding which fixes the s_n and call it the n-iterability embedding.

We will need the notion of Π^1_n iterability for mice with n Woodin cardinals. This notion is a strengthening of the notion of Π^1_2 iterability and the basic can be found in [14]. Π^1_n iterability will be sufficient for comparison of mice with the appropriate number of Woodin cardinals which can be embedded in the background. However the definition of Π^1_n iterability is asymetrical in the case where n is even or odd, reflecting the periodicity phenomenon from descriptive set theory. The definition is slightly more complicated in the case n is odd, and this is the case we are directly concerned with here (Notice that this is the same as Π_n-iterability, where n is even, following Steel's notation, since $\Pi^{HC}_n = \Pi^1_{n+1}$). We refer the reader to [14] for full definitions of Π^1_n iterability.

Using the Spector-Gandy theorem, it is then immediate that the set

$$\{\mathcal{M} : \mathcal{M} \text{ is } \Pi^1_{2n+2} \text{ iterable}\}$$

is a Π^1_{2n+2} set. Steel then shows in [14] that Π^1_{2n+2} iterability is sufficient for comparison of Π^1_{2n+2}-iterable mice with mice which $2n+1$ Woodin cardinals and which are realizable into the background. The reader can consult [14] for a full proof of this fact.

3.2 Inner model characterization of $L[T_{2n}]$

We now state and prove the main theorem of this section.

Theorem 20 *Assume $AD^{L(\mathbb{R})}$. Let T_{2n+2} be the canonical tree which projects to a universal Π^1_{2n+2} set. Then*

$$L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$$

Proof.

Define Steel's tree S_{2n+2} for Π^1_{2n+2}. This will be a tree on $\omega \times \omega \times \omega \times \kappa^1_{2n+3}$. Let $\mathcal{L}$ be the language of premice and let $\mathcal{L}^* = \mathcal{L} \cup \{\dot{a}_i : i < \omega\}$ where the a_i are constants. Let $\langle \varphi_n : n < \omega \rangle$ be a recursive enumeration of the sentence of $\mathcal{L}^*$.

Recall that a theory T is said to be Henkinized if for every formula φ and variable x, there is a constant a such that "$\exists x \varphi(x) \to \varphi(x/a)$" $\in T$, where $\varphi(x/a)$ results from replacing free occurrences of the variable x by the constant a. We say $x \in \mathbb{R}$ codes a premouse if

$$T_x = \{\phi_n : x(n) = 0\}$$

is a complete consistent Henkinized theory of a premouse. If x codes a premouse, we let

$$\mathcal{R}_x = \{\dot{a^x_i} : i < \omega\}$$

be the premouse whose theory is T_x. Define G^- to be the set of triples such that:

1. y codes a C_{2n+2} guided tree $\mathcal{T}_y$ on $\mathcal{M}_{2n+1}^{\#}$
2. z codes a premouse $\mathcal{R}_z$ such that $\mathcal{M}(\mathcal{T}_y) \trianglelefteq \mathcal{R}_z \trianglelefteq L[\mathcal{M}(\mathcal{T}_y)]$ and $\mathcal{R}_z \vDash \text{ZFC}^- +$ "$\delta(\mathcal{T}_y)$ is the largest cardinal"
3. w codes a branch b of $\mathcal{T}_y$ such that $\mathcal{R}_z \trianglelefteq \mathcal{M}_b$

The set G^- is a Δ^1_{2n+2} set. We let

$$G = \{(y,z,w) \in G^- : \text{ either } \mathcal{R}_z \vDash \delta(\mathcal{T}_y) \text{ is not Woodin or } \mathcal{M}(\mathcal{T}_y)^+ \trianglelefteq \mathcal{R}_z\},$$

where $\mathcal{M}(\mathcal{T}_y)^+ = C_{2n+2}(\mathcal{M}(\mathcal{T}_y))$ is the unique suitable premouse extending $\mathcal{M}(\mathcal{T}_y)$ such that $\delta(\mathcal{T}_y)$ is its largest Woodin cardinal. So in G we basically have two cases: the case where $\mathcal{T}_y$ is a short tree and the case where $\mathcal{T}_y$ is a maximal tree. Then the set G is a $\Pi^1_{2n+2}(x)$ set of reals where x codes $\mathcal{M}_{2n+1}^{\#}$.

Define a scale on G as follows. Fix a Σ^1_{2n+2} scale $\vec{\varphi}$ on G^-. Extend $\mathcal{L}^*$ to $\mathcal{L}^{**}$ by introducing new constant symbols $\{\dot{\delta}\} \cup \{\dot{\tau}_i : i < \omega\}$. The intended meaning of the symbols is that if z codes a premouse $\mathcal{R}_z$ which is suitable then we interpret $\dot{\delta}_z$ as the Woodin cardinal of $\mathcal{R}_z$ and $\dot{\tau}_i^z$ as the theories $T_i^{\mathcal{R}_z}$, where i means we only look at the first i indiscernibles. Let $\mathcal{R}^+$ be the $\mathcal{L}^{**}$ structure obtained from $\mathcal{R}_z$. Let $\langle \theta_i : i < \omega \rangle$ be a recursive enumeration of the Σ_0 sentences of $\mathcal{L}^{**}$. Then let

$$T_z^+ = \{\theta_i : \mathcal{R}_z^+ \vDash \theta_i\}$$

Now let

$$\phi_i^0(y,z,w) = 0 \text{ if } \theta_i \in T_z^+ \text{ and } \phi_i^0(y,z,w) = 1 \text{ otherwise.}$$

If $\theta_n = \exists v < \dot{\delta}\psi(v)$ and $\theta_n \in T_z^+$, then we let

$$\phi_n^1(y,z,w) = \text{ least } k \text{ such that } \psi(\dot{a}_k) \in T_z^+$$

and otherwise we let $\phi_n^1(y,z,w) = 0$. Also if $(\dot{a}_k < \gamma_k^{\mathcal{R}_z}) \in T_z^+$ then let

$$\phi_{n,k}^2(y,z,w) = i_{\mathcal{R}_z,\infty}(\dot{a}_n^z)$$

so basically we code the embedding into the norms. Notice, just as in [15], that the first-order theory of $\mathcal{R}^+$ is coded into the norms. The norms also code the elementary embedding $\pi_{\mathcal{R},\infty} \restriction \delta(\mathcal{T}_z)$. Now we code the whole thing as follows: let

$$\phi_{n,m}(y,z,w) = \langle \psi_n(y,z,w), \phi_n^0(y,z,w), \phi_n^1(y,z,w), \phi_{n,m}^2(y,z,w) \rangle$$

Using arguments from Steel one can show that this is a scale [7], see [15]. We actually go ahead and show the following claim:

Claim 6 *$\vec{\phi}_{n,m}$ is a scale on G.*

[7]The key is to show that we have fullness and to use the Dodd-Jensen property.

Proof.

The lower semi-continuity property follows from the Dodd-Jensen property. We refer to Steel [15] for the details. Next we verify the convergence property. So let $(y_n, z_n, w_n) \to (y, z, w)$ with respect to $\vec{\phi}_{n,m}$. We then must see that $(y, z, w) \in G$. Since ψ_n is a scale, then $(y, z, w) \in G^-$. This then implies that $\mathcal{T}_z$ is C_{2n+2}-guided and that we have $\mathcal{R}_z \trianglelefteq \mathcal{M}(\mathcal{T}_z)^+$. Since $(y_n, z_n, w_n) \to (y, z, w)$ with respect to $\vec{\psi}^0$ then we can define $T^+_{z_n} \to T^+$, and T^+ is exists and codes the first-order theory of some unique $\mathcal{P}^+$. Since (y_n, z_n, w_n) converges to (y, z, w) with respect to $\vec{\phi}^1$, then $\mathcal{R}_z = \mathcal{P}$. Next we justify that $\mathcal{P}$ is wellfounded and suitable. For this we use the fact that $\vec{\phi}^2$ is a scale. Let

$$\gamma_n = \sup(\{\xi < \dot{\delta}^{\mathcal{P}^+} : (\xi \text{ is definable over } \mathcal{P} \text{ from } \dot{\tau}_n^{\mathcal{P}^+}\})$$

and let

$$\gamma = \sup_{n<\infty} \gamma_n.$$

Since $\gamma \leq \dot{\delta}^{\mathcal{P}^+} = \delta(\mathcal{T}_y)$ then γ is in the wellfounded part of $\mathcal{P}^+$. Let $\mathcal{P}_1 = \mathcal{H}_1^{\mathcal{P}}(\gamma \cup \{\dot{\tau}_n^{\mathcal{P}^+}\})$ be a Σ_1 Skolem hull which is collapsed on its wellfounded part. Let $\sigma : \mathcal{P}_1 \to \mathcal{P}$ be the canonical embedding Then we must have $crit(\sigma) = \gamma$ by elementarity, so that $\sigma \restriction \gamma = id$. Let $\pi_n : \mathcal{P}_{z_n} \to \mathcal{M}_{2n+1,\infty}$ and define $\pi : \mathcal{P}|\gamma \to \mathcal{M}_{2n+1,\infty}$ by $\pi(\dot{a_j^z}) =$ eventual value of $\pi_n(\dot{a_j^{z_n}})$ as $n \to \infty$. Notice that this eventual value must exist since if $\dot{a_j^z} < \gamma$, then there is $\varphi \in T^+_z$ such that $(\dot{a_j^z} < \gamma) \leftrightarrow \varphi$ and $\varphi \in T^+_{z_n}$ for all sufficiently large n. So there exists a $k < \infty$ such that $\dot{a_j^{z_n}} < \gamma_k^{\mathcal{P}_{z_n}}$. We now extend $\pi : \mathcal{P}|\gamma \to \mathcal{M}_{2n+1,\infty}$ to $\pi : \mathcal{P}_1 \to \mathcal{M}_{2n+1,\infty}$. Notice that this extension need not be an iteration embedding. We also let $\pi(\dot{\tau_n^{\mathcal{P}^+}}) = \dot{\tau_n^{\infty}}$.

Let $c \in \mathcal{P}_1$. Then there exists a $k < \infty$ and a Σ_0 formula φ of the language of premice, and parameters $\dot{a^z_{i_0}}, ..., \dot{a^z_{i_n}} < \gamma_k$ such that

$$c = \text{ the unique } v \text{ s.t } \mathcal{P}|\gamma \vDash \varphi[v, \dot{a^z_{i_0}}, ..., \dot{a^z_{i_n}}, \dot{\tau_n^{\mathcal{P}^+}}]$$

We can do this since $\vec{\phi}^0$ is a scale and since the $T^+_{z_n}$ converge to T^+_z. Then we set

$$\pi(c) = \text{ the unique } v \text{ s.t } \mathcal{M}_{2n+1,\infty}|\gamma_n^{\infty} \vDash \varphi[v, \pi(\dot{a^z_{i_0}}), ..., \pi(\dot{a^z_{i_n}}), \dot{\tau_n^{\infty}}]$$

As usual the map $\pi : \mathcal{P}_1 \to \mathcal{M}_{2n+1,\infty}$ is Σ_1 elementary and welldefined. Now, since by a result of Woodin there exists suitable mice and by [15] we can apply the condensation lemma, then $\gamma = \delta(\mathcal{T}_y)$ as T_y is C_{2n+2} guided. So $\mathcal{P}_1 = \mathcal{P}$ and $\sigma = id$. The other alternative is that $\mathcal{P} \vDash \delta(\mathcal{T}_y)$ is not Woodin because the truth of this statement is kept by all theories $T^+_{z_n}$ then we have that either $\mathcal{R}_z = \mathcal{M}(\mathcal{T}_y)$ or $\mathcal{R}_z \vDash \delta(\mathcal{T}_y)$ is not Woodin so that $G(y, z, w)$ holds.

□

As in [15], one can show that the norms of the above scale are all in $\mathcal{M}^{\#}_{2n+1,\infty}$. The norms of the above scale $\phi_{i,k}$ can be computed to be in for every i in $\Game^{2n+1}\omega(i+1)\text{-}\Pi^1_1$ where we use only the first i indiscernibles, since the theories in i indiscernibles have same complexity $\Game^{2n+1}\omega(i+1)\text{-}\Pi^1_1$, i.e. the types of the first i indiscernibles are exactly $\Game^{2n+1}\omega(i+1)\text{-}\Pi^1_1$. We state the next lemma without proof, a similar computation can be found in [15]:

Lemma 3.5 *For every $i, k < \omega$, the norms $\phi_{i,k}$ of the scale $\vec{\phi}_{n,m}$ are $\Game^{2n+1}\omega(i+1)$-Π^1_1.*

Thus each ϕ_n is $\Delta^1_{2n+1}(x)$. Let S_{2n+2} be the tree from this scale. By the proof of the uniqueness of the $L[T_{2n+2}]$ models we have that $L[T_{2n+2}] = L[S_{2n+2}]$. We'll be done if can show that $L[\mathcal{M}^{\#}_{2n+1,\infty}] = L[S_{2n+2}]$.

First because $\mathcal{M}_{2n+1,\infty}$ is $\Sigma^1_{2n+3}(\mathcal{M}^{\#}_{2n+1})$, then we have that $\mathcal{M}_{2n+1,\infty} \in L[S_{2n+2}] = L[T_{2n+2}]$, since by $\mathcal{Q}$-theory, $\mathcal{M}^{\#}_{2n+1} \in L[T_{2n+2}]$. Letting $i = i_{\mathcal{M}_{2n+1,\infty}} \restriction \delta^{\mathcal{M}_{2n+1}}$ then $i \in L[S_{2n+2}]$ because the iteration embedding i is also $\Sigma^1_{2n+3}(\mathcal{M}^{\#}_{2n+1})$. Thus we have $\mathcal{M}_{2n+1,\infty}, i \in L[S_{2n+2}]$. Hence $\mathcal{M}^{\#}_{2n+1,\infty} \in L[S_{2n+2}]$.

We next show that we have that $L[S_{2n+2}] \subseteq L[\mathcal{M}^{\#}_{2n+1,\infty}]$. Following an idea of Steel (as in [18] or [16] for instance), we build the direct limit tree S. It will be the case that $S \in L[\mathcal{M}^{\#}_{2n+1,\infty}]$ and that Steel's tree S_{2n+2} (and also T_{2n+2}, whichever way we decide to define it) belongs to $L[S]$ by the uniqueness of the $L[T_{2n+2}]$ models. We then define S to be the tree on $\omega \times \omega \times \omega \times \mathcal{M}_{2n+1,\infty}$ of all attempts to build $(x, \pi) \in (\mathbb{R}^3 \times \mathcal{M}^{\omega}_{2n+1,\infty})$ such that

1. x codes the complete theory with parameters of a structure $\mathcal{P}_x$ for the language of premice with universe $\omega \setminus \{0\}$,

2. $\pi(0)$ is a successor cardinal Woodin cutpoint of $\mathcal{P}_x$, and,

3. $\pi \restriction (\omega \setminus \{0\})$ is an elementary embedding from $\mathcal{P}_x$ into $\mathcal{M}_{2n+1,\infty}|\pi(0)$.

Notice that $S_{2n+2} \subseteq S$. It then follows that $S_{2n+2} \in L[S]$ and since $S \in L[\mathcal{M}^{\#}_{2n+1,\infty}]$, we are done.

$\square$

We record the following which now follows from the generalization of the Kechris-Martin theorem, the uniqueness of the $L[T_{2n}]$ models and the above characterization of the $L[T_{2n}]$ in terms of HOD limits of directed systems of mice.

Theorem 21 (Inner model characterization of Π^1_{2n+3}) *Assume $AD^{L(\mathbb{R})}$, let $\delta^{\mathcal{M}_{2n+1,\infty}}$ be the least Woodin cardinal of $\mathcal{M}_{2n+1,\infty}$ and let κ be the least admissible above $\delta^{\mathcal{M}_{2n+1,\infty}}$. Then a set $A \subseteq \mathbb{R}$ is Π^1_{2n+3} if and only if*

$$A(x) \leftrightarrow L_\kappa[\mathcal{M}^{\#}_{2n+1,\infty}, x] \vDash \varphi(x),$$

where $\varphi \in \Sigma_1$.

4 $L[T_{2n}]$, CH and GCH: a proof of a conjecture of Woodin

In this section we give a positive solution to the following problem posed by Woodin:

Conjecture 1 (Woodin) *$L[T_{2n+2}]$ satisfies the GCH for every $n \in \omega$.*

In previous work, see [17] and [18], Steel has shown that assuming AD and Γ-mouse capturing holds, $L[T_\Gamma]$ is an extender model and satisfies the GCH, where Γ is a scaled inductive like pointclass. However recall that in our case Γ is now a non scaled pointclass (i.e. Π^1_{2n} in the case of the projective hierarchy). We would like to thank Sargsyan and especially Woodin, for introducing us to the above conjecture and for discussions on the problem.

We first recall some background of $\mathcal{Q}$-theory. Recall that Q_{2n+3} is a subset of C_{2n+3}, where C_{2n+3} is the largest thin Π^1_{2n+3} set of reals. Also there is a Δ^1_{2n+3}-good wellorder on C_{2n+3} of length $\aleph_1$.

To give some context and for the sake of completeness, we cite the following two theorems of [10].

Theorem 22 (Martin, [10]) *There is a real w such that if $w \in L[T_{2n+1}, x]$ then*

$$\mathbb{R} \cap HOD^{L[T_{2n+1},x]} = Q_{2n+3}$$

The next theorem, due to Woodin, shows that relativizing to a real is the same as adjoining a real to HOD.

Theorem 23 (Woodin, [10]) *For every real w there is a real z such that if $w, z \in L[T_{2n+1}, x]$ then $\mathbb{R} \cap HOD^{L[T_{2n+1},x]}_{T_{2n+1}}[w] = \mathbb{R} \cap HOD^{L[x]}_{T_{2n+1},w} = Q_{2n+3}$*

What will help in correctly identifying $L[T_{2n+2}]$ from the point of view of inner model theory is a characterization of the reals of $L[T_{2n+2}]$. We show the following theorem:

Theorem 24 *(The reals of $L[T_{2n+2}]$)*

Let Q_{2n+3} be the largest bounded Π^1_{2n+3} set of reals and let y_{2n+3} be the least nontrivial Π^1_{2n+3} singleton and let $y_{2n+3}(x)$ be the least nontrivial $\Pi^1_{2n+3}(x)$ singleton. Let $\mathcal{Y}_{2n+3} = Q_{2n+3} \cup \{y_{2n+3}\} \cup \{y_{2n+3}(x) : x \in Q_{2n+3}\}$. Therefore $L[T_{2n+2}]$ is y_{2n+1}-closed and $\mathbb{R} \cap L[T_{2n+2}] = \mathcal{Y}_{2n+3}$.

Proof.

$L[T_{2n+2}]$ can compute left most branch of a Δ^1_{2n+3} scale on a Δ^1_{2n+3} set of reals and it is a result of Harrington that the real from the left most branch of the tree from this scale, provided the set $A \in \Delta^1_{2n+3}$ on which we put the scale, does not contain any Δ^1_{2n+3} real, is $\Delta^1_{2n+3}(\mathcal{M}^{\#}_{2n+1})$ and vice-versa. So the least non trivial Π^1_{2n+3} singleton is in $L[T_{2n+2}]$. Next, using the generalizations of the Kechris-Martin theorem to all pointclasses Π^1_{2n+3}, one can show [8] that $Q_{2n+3} \subseteq L[T_{2n+2}]$, so $L[T_{2n+2}]$ can also compute the left most real of the tree of a $\Delta^1_{2n+3}(x)$ scale on a $\Delta^1_{2n+3}(x)$ set of reals, for every $x \in Q_{2n+3}$. So $y_{2n+3}(x) \in L[T_{2n+2}]$ for every $x \in Q_{2n+3}$.

□

The above theorems of Martin and Woodin suggest strongly that

$$\mathrm{HOD}^{L[T_{2n+1},x]}_{T_{2n+1}} \cap V_{\kappa^1_{2n+3}} = \mathcal{M}_{2n+1,\infty}$$

[8] This is shown in the author's PhD thesis, see [1], section 3.4 for a generalization of the Kechris-Martin theorem and section 3.5 for a proof that $Q_{2n+3} \subseteq L[T_{2n+2}]$.

and that

$$\mathrm{HOD}_{T_{2n+1}}^{L[T_{2n+1},x]} = L[\mathcal{M}_{2n+1,\infty}, \Sigma_0],$$

where Σ_0 is a certain fragment of the full iteration strategy Σ on $\mathcal{M}_{2n+1}$, see [18].

As mentioned above, recall that for $\alpha = \underset{\sim}{\delta}^1_{2n+1}$ then we have that $L[T_{2n+1}] \cap V_{\underset{\sim}{\delta}^1_{2n+1}}$ is an iterate of a $\mathcal{M}_{2n}$ cut a the least strong cardinal to its least Woodin cardinal and the height of that iterate is exactly $\underset{\sim}{\delta}^1_{2n+1}$, since $\underset{\sim}{\delta}^1_{2n+1}$ is the least strong to the bottom Woodin δ_∞ in the direct limit of all iterates of $\mathcal{M}_{2n}$. We recall how this computation takes place. The set up below is from [18].

Definition 4.1 *A premouse $\mathcal{P}$ is Γ-properly small iff $\mathcal{P}$ is countable, has a largest cardinal which is a cutpoint of $\mathcal{P}$ and for every $\eta < o(\mathcal{P})$,*

1. $Lp^\Gamma(\mathcal{P}|\eta) \trianglelefteq \mathcal{P}$,
2. $Lp^\Gamma(\mathcal{P}|\eta) \vDash \eta$ *is not a Woodin cardinal,*
3. *If η is a strong cutpoint of $\mathcal{P}$, then $Lp^\Gamma(\mathcal{P}|\eta) = \mathcal{P}|(\eta^+)^{\mathcal{P}}$.*

If Σ is the $(\omega, \omega_1, \omega_1)$ strategy of $\mathcal{P}$ given by the above then we say that it is Lp^Γ guided and the non-dropping iterates of $\mathcal{P}$ via Σ are Γ properly small. Σ is unique and has by the Dodd-Jensen property. This allows defining the direct limit of all non-dropping Lp^Γ guided iterates of $\mathcal{P}$. So let $\mathcal{I} = \{\mathcal{P} : \mathcal{P}$ is Γ-properly small and Γ-correctly iterable$\}$. For $\mathcal{P}, \mathcal{Q} \in \mathcal{I}$, we let

$$\mathcal{P} \prec \mathcal{Q} \leftrightarrow \exists \eta \text{ s.t } \eta \text{ is a strong cutpoint of } \mathcal{Q}, \mathcal{Q}|\eta \text{ is a } \Gamma\text{-correct iterate of } \mathcal{P}$$

It is then shown in [18] using a comparison argument that the system $(\mathcal{I}, \preceq)$ is a directed system of mice, and thus by the Dodd-Jensen property, the direct limit of this system, $\mathcal{M}_\infty$ is well-defined, wellfounded and that $\mathcal{M}_\infty = L[T_\Gamma]$.

We now turn to the proof of the GCH in the models $L[T_{2n}]$. We are grateful to Hugh Woodin for guiding us to show the main theorem of this section. Following an idea of Hugh Woodin, we first show that the GCH holds in $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$. Then the GCH will hold in $L[T_{2n+2}]$ using a usual Gödel condensation argument for relative constructibility, see for example [5] chapter 19. The goal is then to show that $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$ is a direct limit of fully sound structures. As in the theorem in the previous section, we will then show that $L[T_{2n+2}] = L[\mathcal{M}^\#]$ for some $\mathcal{M}$ which is a direct limit of fully sound structures and such that $L[\mathcal{M}^\#] \cap V_{\kappa^1_{2n+3}} = \mathcal{M}$. So we will require that $o(\mathcal{M}) = \kappa^1_{2n+3}$. We start with the following definition:

Definition 4.2 ($\mathcal{M}^\#_{2n+1}$-closed mouse) *Let $\mathcal{M}$ be a premouse. Then we say that $\mathcal{M}$ is a $\mathcal{M}^\#_{2n+1}$-closed premouse if for every $A \in \mathcal{M}$, we have $\mathcal{M}^\#_{2n+1}(A) \in \mathcal{M}$. Also, $\mathcal{M}$ is a $\mathcal{M}^\#_{2n+1}$-closed mouse if it is a $\mathcal{M}$ is a $\mathcal{M}^\#_{2n+1}$-closed premouse and has an $(\omega, \omega_1, \omega_1)$-iteration strategy Σ.*

Next we need to define the Woodin mice which will constitute our directed system below.

Definition 4.3 *We say $\mathcal{N}$ is a n-Woodin mouse if the following conditions are satisfied:*

1. $\mathcal{N} = L(\mathcal{N})^{\#} \cap V_\delta$, *where* $\delta = o(\mathcal{N})$,
2. $L(\mathcal{N}) \vDash \delta$ *is a Woodin cardinal .*
3. $\mathcal{N}$ *has* n *Woodin cardinals.*

We next define the iteration strategy of an n-Woodin mouse in the case n is odd.

Definition 4.4 (Iterability for n-Woodin mice) *Let $\mathcal{N}$ be an n-Woodin mouse. We say $\mathcal{N}$ is correctly iterable if whenever $\vec{\mathcal{T}}$ is a countable stack of C_{2n+2} guided normal trees of successor lengths on $\mathcal{N}$ with last model $\mathcal{Q}$, then*

1. *$\mathcal{Q}$ is wellfounded and if the branch from $\mathcal{N}$ to $\mathcal{Q}$ of $\vec{\mathcal{T}}$ does not drop, then $\mathcal{Q}$ is an n-Woodin mouse and*
2. *If $\mathcal{U}$ is a C_{2n+2} guided normal tree on $\mathcal{Q}$ then*
   (a) *$\mathcal{U}$ is a short tree and*
   (b) *If $\mathcal{U}$ has a last model then it can be freely extended by one more ultrapower that is every putative normal iteration tree $\mathcal{T}$ extending $\mathcal{U}$ and having length $lh(\mathcal{U})+1$ has a wellfounded last model and moreover this last model is an n-Woodin mouse if the leading branch does not drop, and*
   (c) *If $\mathcal{U}$ has limit length then $\mathcal{U}$ has a cofinal wellfounded branch b such that $\mathcal{Q}(b, U) = \mathcal{Q}(\mathcal{U})$ and $\mathcal{M}_b^{\mathcal{U}}$ is an n-Woodin mouse if the branch from $\mathcal{N}$ to $\mathcal{Q}$ to $\mathcal{M}_b^{\mathcal{U}}$ does not drop.*

By Steel, see [14], the above notion of iterability for n-Woodin mice is equivalent to Π^1_{2n+2} iterability, defined above. Let $\mathcal{N}$ be the least $2n+1$-Woodin mouse, that is if $\mathcal{S} \lhd \mathcal{N}$ then $\mathcal{S}$ fails one of the conditions above. Let $\Sigma_{\mathcal{N}}$ be the iteration strategy of $\mathcal{N}$. Define

$$\mathcal{I} = \{\mathcal{P} : \mathcal{P} \text{ is a } \Sigma\text{-iterate of } \mathcal{N}\}$$

and for $\mathcal{P}, \mathcal{Q} \in \mathcal{I}$, we let

$$\mathcal{P} \prec^* \mathcal{Q} \leftrightarrow \exists \eta(\eta \text{ is a Woodin cardinal cutpoint of } \mathcal{Q} \text{ and } \mathcal{Q}|\eta \text{ is a countable } \Sigma\text{-iterate of } \mathcal{P})$$

Then notice that $(\mathcal{I}, \prec^*)$ is a partial order.

Lemma 4.5 $(\mathcal{I}, \prec^*)$ *is countably directed.*

The proof of the above is as usual and we chose to omit it. The proof is given in [17].

Let now $\mathcal{N}_\infty$ be the direct limit of the system $(\mathcal{I}, \prec^*)$. Then since $(\mathcal{I}, \prec^*)$ is countably directed, $\mathcal{N}_\infty$ is wellfounded. $\mathcal{N}_\infty$ is the direct limit of all countable iterates of the least $\mathcal{N}$ satisfying the above two conditions, and we can define this direct limit by the Dodd-Jensen property of the $\Sigma_{\mathcal{N}}$. Notice that $\mathcal{N}_\infty$ is itself a countable iterate of $\mathcal{N}$ via $\Sigma_{\mathcal{N}}$. It then follows by the proof in the above section that

$$L[T_{2n+2}] = L[\mathcal{N}_\infty^\#],$$

since the iteration strategy Σ_∞ of $\mathcal{N}_\infty$ is Π^1_{2n+3}. Notice that

$$\mathcal{N}_\infty = L[\mathcal{N}_\infty^\#] \cap V_{\delta_\infty} = L[\mathcal{N}_\infty^\#] \cap V_{\kappa^1_{2n+3}} = L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}.$$

Therefore $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$ is a direct limit of all Σ iterates of $\mathcal{N}$. Since $\mathcal{N}_\infty$ is fully sound then $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}} \models$ GCH. Then by a condensation argument as in the Gödel condensation lemma, $L[T_{2n+2}] \models$ GCH. See [5] for the condensation lemma in the context of relative constructibility.

We drop down to the case of $\mathcal{M}_1^\#$ for the moment. It then remains to show that $\mathcal{N}_\infty$ is $\mathcal{M}_1^\#$-closed and we finish by showing the following lemma. So $\mathcal{N}_\infty$ is the least active mouse closed under $\mathcal{M}_1^\#$ which projects to ω. It is sometimes referred to in the literature as $\mathcal{M}_1^{\#^\#}$.

Lemma 4.6 *$\mathcal{N}_\infty$ is $\mathcal{M}_1^\#$-closed. Therefore $\mathcal{N}_\infty$ does not project at or below δ_∞, $\mathcal{N}_\infty$ is fully sound and*

$$\rho_\omega(\mathcal{N}_\infty) > o(\mathcal{N}_\infty) = \delta_\infty.$$

Proof.

Suppose not and let $A \in \mathcal{N}_\infty$ such that $\mathcal{M}_1^\#(A) \notin \mathcal{N}_\infty$. Let $\mathcal{P} \in \mathcal{I}$ be a countable iterate of $\mathcal{N}$ such that $\pi_{\mathcal{P},\infty} : \mathcal{P} \to \mathcal{N}_\infty$ is the iteration embedding. Let $\pi : L(\mathcal{P}) \to L(\mathcal{N}_\infty)$ be elementary such that $\pi|\mathcal{P} = \pi_{\mathcal{P},\infty}$ and such that $\delta_\infty, \mathcal{N}_\infty, \mathcal{P}$ and $A \in ran(\pi)$. Let $\bar{A} \in \mathcal{P}$ such that $\pi(\bar{A}) = A$. Notice that $\mathcal{M}_1^\#(\bar{A})$ has same size as $\bar{A}$. It then follows it is a bounded subset of $\delta^{\mathcal{P}}$. Since the $\mathcal{M}_1^\#$ operator condenses well then we have that $\mathcal{M}_1^\#(\pi^{-1}(A)) = \pi^{-1}(\mathcal{M}_1^\#(A))$. So $\mathcal{M}_1^\#(\pi^{-1}(A)) \notin \mathcal{P}$. But then $L(\mathcal{P}) \not\models \delta^{\mathcal{P}}$ is Woodin . Contradiction.

□

The above can be generalized in the obvious way to all $\mathcal{M}_{2n+1}^\#$. It then follows that $L[T_{2n}] \models$ GCH. From the above it should now be possible to adapt the standard proofs that $\square_\kappa$ for $\kappa > \aleph_1$ a cardinal to show that if $V = L[T_{2n}]$ then for any cardinal $\kappa > \aleph_1$, $\square_\kappa$ holds. Using purely inner model theoretic tools and $\mathcal{Q}$-theory for inductive pointclasses, it should be possible to push the analysis to pointclasses higher than those of the projective hierarchy. Or it may as well be possible that the very fine analysis of $L(\mathbb{R})$ of Jackson is necessary to carry this analysis further.

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